Annals of Mathematics, **157** (2003), 1–43

Axiom A maps are dense in the space of unimodal maps in the C^k topology

By O. S. Kozlovski

Abstract

In this paper we prove C^k structural stability conjecture for unimodal maps. In other words, we shall prove that Axiom A maps are dense in the space of C^k unimodal maps in the C^k topology. Here k can be $1, 2, \dots, \infty, \omega$.

1. Introduction

1.1. *The structural stability conjecture.* The structural stability conjecture was and remains one of the most interesting and important open problems in the theory of dynamical systems. This conjecture states that a dynamical system is structurally stable if and only if it satisfies Axiom A and the transversality condition. In this paper we prove this conjecture in the simplest nontrivial case, in the case of smooth unimodal maps. These are maps of an interval with just one critical turning point.

To be more specific let us recall the definition of Axiom A maps:

Definition 1.1. Let X be an interval. We say that a C^k map $f : X \hookleftarrow$ satisfies the *Axiom* A conditions if:

- f has finitely many hyperbolic periodic attractors,
- the set $\Sigma(f) = X \setminus \mathfrak{B}(f)$ is hyperbolic, where $\mathfrak{B}(f)$ is a union of the basins of attracting periodic points.

This is more or less a classical definition of the Axiom A maps; however in the case of C^2 one-dimensional maps Mañè has proved that a C^2 map satisfies Axiom A if and only if all its periodic points are hyperbolic and the forward iterates of all its critical points converge to some periodic attracting points.

It was proved many years ago that Axiom A maps are C^2 structurally stable if the critical points are nondegenerate and the "no-cycle" condition is fulfilled (see, for example, [dMvS]). However the opposite question "Does

structural stability imply Axiom A?" appeared to be much harder. It was conjectured that the answer to this question is affirmative and it was assigned the name "structural stability conjecture". So, the main result of this paper is the following theorem:

THEOREM A. *Axiom* A *maps are dense in the space of* $C^\omega(\Delta)$ *unimodal maps in the* $C^\omega(\Delta)$ *topology* (Δ *is an arbitrary positive number*).

Here $C^\omega(\Delta)$ denotes the space of real analytic functions defined on the interval which can be holomorphically extended to a Δ-neighborhood of this interval in the complex plane.

Of course, since analytic maps are dense in the space of smooth maps it immediately follows that C^k unimodal Axiom A maps are dense in the space of all unimodal maps in the C^k topology, where $k = 1, 2, \ldots, \infty$.

This theorem, together with the previously mentioned theorem, clearly implies the structural stability conjecture:

THEOREM B. *A* C^k *unimodal map* f *is* C^k *structurally stable if and only if the map* f *satisfies the Axiom* A *conditions and its critical point is nondegenerate and nonperiodic,* $k = 2, \ldots, \infty, \omega$.[1]

Here the critical point is called *nondegenerate* if the second derivative at the point is not zero.

In this theorem the number k is greater than one because any unimodal map can be C^1 perturbed to a nonunimodal map and, hence, there are no C^1 structurally stable unimodal maps (the topological conjugacy preserves the number of turning points). For the same reason the critical point of a structurally stable map should be nondegenerate.

In fact, we will develop tools and techniques which give more detailed results. In order to formulate them, we need the following definition: The map f is *regular* if either the ω-limit set of its critical point c does not contain neutral periodic points or the ω-limit set of c coincides with the orbit of some neutral periodic point. For example, if the map has negative Schwarzian derivative, then this map is regular. Regular maps are dense in the space of all maps (see Lemma 4.7). We will also show that if the analytic map f does not have neutral periodic points, then this map can be included in a family of regular analytic maps.

THEOREM C. *Let* X *be an interval and* $f_\lambda : X \hookleftarrow$ *be an analytic family of analytic unimodal regular maps with a nondegenerate critical point,* $\lambda \in \Omega \subset \mathbb{R}^N$ *where* Ω *is a open set. If the family* f_λ *is nontrivial in the sense that there exist two maps in this family which are not combinatorially*

[1] If $k = \omega$, then one should consider the space $C^\omega(\Delta)$.

equivalent, then Axiom A *maps are dense in this family. Moreover, let* Υ_{λ_0} *be a subset of* Ω *such that the maps* f_{λ_0} *and* $f_{\lambda'}$ *are combinatorially equivalent for* $\lambda' \in \Upsilon_{\lambda_0}$ *and the iterates of the critical point of* f_{λ_0} *do not converge to some periodic attractor. Then the set* Υ_{λ_0} *is an analytic variety. If* $N = 1$*, then* $\Upsilon_{\lambda_0} \cap Y$*, where the closure of the interval* Y *is contained in* Ω*, has finitely many connected components.*

Here we say that two unimodal maps f and $\hat{f}$ are *combinatorially equivalent* if there exists an order-preserving bijection $h : \cup_{n\geq 0} f^n(c) \to \cup_{v\geq 0}\hat{f}(\hat{c})$ such that $h(f^n(c)) = \hat{f}^n(\hat{c})$ for all $n \geq 0$, where c and $\hat{c}$ are critical points of f and $\hat{f}$. In the other words, f and $\hat{f}$ are combinatorially equivalent if the order of their forward critical orbit is the same. Obviously, if two maps are topologically conjugate, then they are combinatorially equivalent.

Theorem A gives only global perturbations of a given map. However, one can want to perturb a map in a small neighborhood of a particular point and to obtain a nonconjugate map. This is also possible to do and will be considered in a forthcoming paper. (In fact, all the tools and strategy of the proof will be the same as in this paper.)

1.2. *Acknowledgments.* First and foremost, I would like to thank S. van Strien for his helpful suggestions, advice and encouragement. Special thanks go to W. de Melo who pointed out that the case of maps having neutral periodic points should be treated separately. His constant feedback helped to improve and clarify the presentation of the paper.

G. Świątek explained to me results on the quadratic family and our many discussions clarified many of the concepts used here. J. Graczyk, G. Levin and M. Tsuji gave me helpful feedback at talks that I gave during the International Congress on Dynamical Systems at IMPA in Rio de Janeiro in 1997 and during the school on dynamical systems in Toyama, Japan in 1998. I also would like to thank D.V. Anosov, M. Lyubich, D. Sands and E. Vargas for their useful comments.

This work has been supported by the Netherlands Organization for Scientific Research (NWO).

1.3. *Historical remarks.* The problem of the description of the structurally stable dynamical systems goes back to Poincaré, Fatou, Andronov and Pontrjagin. The explicit definition of a structurally stable dynamical system was first given by Andronov although he assumed one extra condition: the C^0 norm of the conjugating homeomorphism had to tend to 0 when ϵ goes to 0.

Jakobson proved that Axiom A maps are dense in the C^1 topology, [Jak]. The C^2 case is much harder and only some partial results are known. Blokh and Misiurewicz proved that any map satisfying the Collect-Eckmann conditions can be C^2 perturbed to an Axiom A map, [BM2]. In [BM1] they extend

this result to a larger class of maps. However, this class does not include the infinitely renormalizable maps, and it does not cover nonrenormalizable maps completely.

Much more is known about one special family of unimodal maps: quadratic maps $Q_c : x \mapsto x^2 + c$. It was noticed by Sullivan that if one can prove that if two quadratic maps Q_{c_1} and Q_{c_2} are topologically conjugate, then these maps are quasiconformally conjugate, then this would imply that Axiom A maps are dense in the family Q. Now this conjecture is completely proved in the case of real c and many people made contributions to its solution: Yoccoz proved it in the case of the finitely renormalizable quadratic maps, [Yoc]; Sullivan, in the case of the infinitely renormalizable unimodal maps of "bounded combinatorial type", [Sul1], [Sul2]. Finally, in 1992 there appeared a preprint by Świątek where this conjecture was shown for all real quadratic maps. Later this preprint was transformed into a joint paper with Graczyk [GS]. In the preprint [Lyu2] this result was proved for a class of quadratic maps which included the real case as well as some nonreal quadratic maps; see also [Lyu4]. Another proof was recently announced in [Shi]. Thus, the following important rigidity theorem was proved:

Theorem (Rigidity Theorem). *If two quadratic non Axiom* A *maps* Q_{c_1} *and* Q_{c_2} *are topologically conjugate* $(c_1, c_2 \in \mathbb{R})$, *then* $c_1 = c_2$.

1.4. *Strategy of the proof.* Thus, we know that we can always perturb a quadratic map and change its topological type if it is not an Axiom A map. We want to do the same with an arbitrary unimodal map of an interval. So the first reasonable question one may ask is "What makes quadratic maps so special"? Here is a list of major properties of the quadratic maps which the ordinary unimodal maps do not enjoy:

- Quadratic maps are analytic and they have nondegenerate critical point;
- Quadratic maps have negative Schwarzian derivative;
- Inverse branches of quadratic maps have "nice" extensions to the complex plane (in terminology which we will introduce later we will say that the quadratic maps belong to the Epstein class);
- Quadratic maps are polynomial-like maps;
- The quadratic family is rigid in the sense that a quasiconformal conjugacy between two non Axiom A maps from this family implies that these maps coincide;
- Quadratic maps are regular.

We will have to compensate for the lack of these properties somehow.

First, we notice that since the analytic maps are dense in the space of C^k maps it is sufficient to prove the C^k structural stability conjecture only for analytic maps, i.e., when k is ω. Moreover, by the same reasoning we can assume that the critical point of a map we want to perturb is nondegenerate.

The negative Schwarzian derivative condition is a much more subtle property and it provides the most powerful tool in one-dimensional dynamics. There are many theorems which are proved only for maps with negative Schwarzian derivative. However, the tools described in [Koz] allow us to forget about this condition! In fact, any theorem proved for maps with negative Schwarzian derivative can be transformed (maybe, with some modifications) in such a way that it is not required that the map have negative Schwarzian derivative anymore. Instead of the negative Schwarzian derivative the map will have to have a nonflat critical point.

In the first versions of this paper, to get around the Epstein class, we needed to estimate the sum of lengths of intervals from an orbit of some interval. This sum is small if the last interval in the orbit is small. However, Lemma 2.4 in [dFdM] allows us to estimate the shape of pullbacks of disks if one knows an estimate on the sum of lengths of intervals in some power greater than 1. Usually such an estimate is fairly easy to arrive at and in the present version of the paper we do not need estimates on the sum of lengths any more.

Next, the renormalization theorem will be proved; i.e. we will prove that for a given unimodal analytical map with a nondegenerate critical point there is an induced holomorphic polynomial-like map, Theorem 3.1. For infinitely renormalizable maps this theorem was proved in [LvS]. For finitely renormalizable maps we will have to generalize the notion of polynomial-like maps, because one can show that the classical definition does not work in this case for all maps.

Finally, using the method of quasiconformal deformations, we will construct a perturbation of any given analytic regular map and show that any analytic map can be included in a nontrivial analytic family of unimodal regular maps.

If the critical point of the unimodal map is not recurrent, then either its forward iterates converge to a periodic attractor (and if all periodic points are hyperbolic, the map satisfies Axiom A) or this map is a so-called Misiurewicz map. Since in the former case we have nothing to do the only interesting case is the latter one. However, the Misiurewicz maps are fairly well understood and this case is really much simpler than the case of maps with a recurrent critical point. So, usually we will concentrate on the latter, though the case of Misiurewicz maps is also considered.

We have tried to keep the exposition in such a way that all section of the paper are as independent as possible. Thus, if the reader is interested only in

the proofs of the main theorems, believes that maps can be renormalized as described in Theorem 3.1 and is familiar with standard definitions and notions used in one-dimensional dynamics, then he/she can start reading the paper from Section 4.

1.5. *Cross-ratio estimates.* Here we briefly summarize some known facts about cross-ratios which we will use intensively throughout the paper.

There are several types of cross-ratios which work more or less in the same way. We will use just a standard cross-ratio which is given by the formula:

$$\mathrm{b}(T,J)=\frac{|J||T|}{|T^-||T^+|}$$

where $J\subset T$ are intervals and T^-, T^+ are connected components of $T\setminus J$.

Another useful cross-ratio (which is in some sense degenerate) is the following:

$$\mathrm{a}(T,J)=\frac{|J||T|}{|T^-\cup J||J\cup T^+|}$$

where the intervals T^- and T^+ are defined as before.

If f is a map of an interval, we will measure how this map distorts the cross-ratios and introduce the following notation:

$$\begin{aligned}\mathrm{B}(f,T,J) &= \frac{\mathrm{b}(f(T),f(J))}{\mathrm{b}(T,J)}\\ \mathrm{A}(f,T,J) &= \frac{\mathrm{a}(f(T),f(J))}{\mathrm{a}(T,J)}.\end{aligned}$$

It is well-known that maps having negative Schwarzian derivative increase the cross-ratios: $\mathrm{B}(f,T,J)\geq 1$ and $\mathrm{A}(f,T,J)\geq 1$ if $J\subset T$, $f|_T$ is a diffeomorphism and the C^3 map f has negative Schwarzian derivative. It turns out that if the map f does not have negative Schwarzian derivative, then we also have an estimate on the cross-ratios provided the interval T is small enough. This estimate is given by the following theorems (see [Koz]):

THEOREM 1.1. *Let $f: X \hookleftarrow$ be a C^3 unimodal map of an interval to itself with a nonflat nonperiodic critical point and suppose that the map f does not have any neutral periodic points. Then there exists a constant $C_1>0$ such that if M and I are intervals, I is a subinterval of M, $f^n|_M$ is monotone and $f^n(M)$ does not intersect the immediate basins of periodic attractors, then*

$$\begin{aligned}\mathrm{A}(f^n,M,I) &> \exp(-C_1\,|f^n(M)|^2),\\ \mathrm{B}(f^n,M,I) &> \exp(-C_1\,|f^n(M)|^2).\end{aligned}$$

Fortunately, we will usually deal only with maps which have no neutral periodic points because such maps are dense in the space of all unimodal maps. However, at the end we will need some estimates for maps which do have neutral periodic points and then we will use another theorem ([Koz]):

THEOREM 1.2. *Let $f : X \hookleftarrow$ be a C^3 unimodal map of an interval to itself with a nonflat nonperiodic critical point. Then there exists a nice*[2] *interval T such that the first entry map to the interval $f(T)$ has negative Schwarzian derivative.*

1.6. *Nice intervals and first entry maps.* In this section we introduce some definitions and notation.

The *basin* of a periodic attracting orbit is a set of points whose iterates converge to this periodic attracting orbit. Here the periodic attracting orbit can be neutral and it can attract points just from one side. The *immediate* basin of a periodic attractor is a union of connected components of its basin whose contain points of this periodic attracting orbit. The union of immediate basins of all periodic attracting points will be called the *immediate basin of attraction* and will be denoted by $\mathfrak{B}_0$.

We say that the point x' is *symmetric* to the point x if $f(x) = f(x')$. In this case we call the interval $[x, x']$ *symmetric* as well. A symmetric interval I around a critical point of the map f is called *nice* if the boundary points of this interval do not return into the interior of this interval under iterates of f. It is easy to check that there are nice intervals of arbitrarily small length if the critical point is not periodic.

Let $T \subset X$ be a nice interval and $f : X \hookleftarrow$ be a unimodal map. $R_T : U \to T$ denotes the first entry map to the interval T, where the open set U consists of points which occasionally enter the interval T under iterates of f. If we want to consider the first return map instead of the first entry map, we will write $R_T|_T$. If a connected component J of the set U does not contain the critical point of f, then $R_T : J \to T$ is a diffeomorphism of the interval J onto the interval T. A connected component of the set U will be called a *domain* of the first entry map R_T, or a *domain* of the nice interval T. If J is a domain of R_T, the map $R_T : J \to T$ is called a *branch* of R_T. If a domain contains the critical point, it is called *central*.

Let T_0 be a small nice interval around the critical point c of the map f. Consider the first entry map R_{T_0} and its central domain. Denote this central domain as T_1. Now we can consider the first entry map R_{T_1} to T_1 and denote its central domain as T_2 and so on. Thus, we get a sequence of intervals $\{T_k\}$ and a sequence of the first entry maps $\{R_{T_k}\}$.

[2]The definition of nice intervals is given in the next subsection.

We will distinguish several cases. If $c \in R_{T_k}(T_{k+1})$, then R_{T_k} is called a *high* return and if $c \notin R_{T_k}(T_{k+1})$, then R_{T_k} is a *low* return. If $R_{T_k}(c) \in T_{k+1}$, then R_{T_k} is a *central* return and otherwise it is a *noncentral* return.

The sequence $T_0 \supset T_1 \supset \cdots$ can converge to some nondegenerate interval $\tilde{T}$. Then the first return map $R_{\tilde{T}}|_{\tilde{T}}$ is again a unimodal map which we call a *renormalization* of f and in this case the map f is called *renormalizable* and the interval $\tilde{T}$ is called a *restrictive* interval. If there are infinitely many intervals such that the first return map of f to any of these intervals is unimodal, then the map f is called *infinitely renormalizable*.

Suppose that $g : X \hookleftarrow$ is a C^1 map and suppose that $g|_J : J \to T$ is a diffeomorphism of the interval J onto the interval T. If there is a larger interval $J' \supset J$ such that $g|_{J'}$ is a diffeomorphism, then we will say that the range of the map $g|_J$ can be *extended* to the interval $g(J')$.

We will see that any branch of the first entry map can be decomposed as a quadratic map and a map with some definite extension.

LEMMA 1.1. *Let f be a unimodal map, T be a nice interval, J be its central domain and V be a domain of the first entry map to J which is disjoint from J, i.e. $V \cap J = \emptyset$. Then the range of the map $R_J : V \to J$ can be extended to T.*

This is a well-known lemma; see for example [dMvS] or [Koz].

We say that an interval T is a τ-scaled neighborhood of the interval J, if T contains J and if each component of $T \setminus J$ has at least length $\tau|J|$.

2. Decay of geometry

In this section we state an important theorem about the exponential "decay of geometry". We will consider *unimodal nonrenormalizable* maps with a recurrent *quadratic* critical point. It is known that in the multimodal case or in the case of a degenerate critical point this theorem does not hold.

Consider a sequence of intervals $\{T_0, T_1, \ldots\}$ such that the interval T_0 is nice and the interval T_{k+1} is a central domain of the first entry map R_{T_k}. Let $\{k_l,\, l = 0, 1, \ldots\}$ be a sequence such that T_{k_l} is a central domain of a noncentral return. It is easy to see that since the map f is nonrenormalizable the sequence $\{k_l\}$ is unbounded and the size of the interval T_k tends to 0 if k tends to infinity.

The decay of the ratio $\frac{|T_{k_l+1}|}{|T_{k_l}|}$ will play an important role in the next section.

THEOREM 2.1. *Let f be an analytic unimodal nonrenormalizable map with a recurrent quadratic critical point and without neutral periodic points. Then the ratio $\frac{|T_{k_l+1}|}{|T_{k_l}|}$ decays exponentially fast with l.*

This result was suggested in [Lyu3] and it has been proven in [GS] and [Lyu4] in the case when the map is quadratic or when it is a box mapping. To be precise we will give the statement of this theorem below, but first we introduce the notion of a box mapping.

Definition 2.1. Let $A \subset \mathbb{C}$ be a simply connected Jordan domain, $B \subset A$ be a domain each of whose connected components is a simply connected Jordan domain and let $g : B \to A$ be a holomorphic map. Then g is called *a holomorphic box mapping* if the following assumptions are satisfied:

- g maps the boundary of a connected component of B onto the boundary of A,
- There is one component of B (which we will call *a central domain*) which is mapped in the 2-to-1 way onto the domain A (so that there is a critical point of g in the central domain),
- All other components of B are mapped univalently onto A by the map g,
- The iterates of the critical point of g never leave the domain B.

In our case all holomorphic box mappings will be called *real* in the sense that the domains B and A are symmetric with respect to the real line and the restriction of g onto the real line is real.

We will say that a real holomorphic box mapping F is *induced* by an analytic unimodal map f if any branch of F has the form f^n.

We can repeat all constructions we used for a real unimodal map in the beginning of this section for a real holomorphic box mapping. Denote the central domain of the map g as A_1 and consider the first return map onto A_1. This map is again a real holomorphic box mapping and we can again consider the first return map onto the domain A_2 (which is a central domain of the first entry map onto A_1) and so on. The definition of the central and noncentral returns and the definition of the sequence $\{k_l\}$ can be literally transferred to this case if g is nonrenormalizable (this means that the sequence $\{k_l\}$ is unbounded).

Theorem 2.2 ([GS], [Lyu4]). *Let $g : B \to A$ be a real holomorphic nonrenormalizable box mapping with a recurrent critical point and let the modulus of the annulus $A \setminus \hat{B}$ be uniformly bounded from* 0, *where $\hat{B}$ is any connected component of the domain B. Then the ratio $\frac{|A_{k_l+1}|}{|A_{k_l}|}$ tends to* 0 *exponentially fast, where $|A_k|$ is the length of the real trace of the domain A_k.*

Here the real trace of the domain is just the intersection of this domain with the real line.

So, if we can construct an induced box mapping, we will be able to prove Theorem 2.1. Fortunately, this construction has been done in [LvS] and in the less general case in [GS], [Lyu3].

THEOREM 2.3. *For any analytic unimodal map f with a nondegenerate critical point there exists an induced holomorphic box mapping $F : B \to A$. Moreover, there exists a constant $C > 0$ such that if $\hat{B}$ is a connected component of B, then* $\operatorname{mod}(A \setminus \hat{B}) > C$.

In fact, this theorem was proven in [LvS] for infinitely renormalizable maps in full generality and for the finitely renormalizable maps satisfying two extra assumptions: f has negative Schwarzian derivative and f belongs to the Epstein class (for definition of the Epstein class see Appendix 5.2). However, these conditions are not necessary any more. Indeed, Theorem 2.3 is a consequence of some estimates (usually called “complex bounds”). In [LvS] these estimates are robust in the following sense: if you change all constants involved by some spoiling factor which is close to 1, then the estimates still remain true. Now, according to [Koz] on small scales one has the cross-ratio estimates as in the case of maps with negative Schwarzian derivative, but with some spoiling factor close to 1 (see Theorems 1.1 and 1.2). Lemma 2.4 in [dFdM] gives estimates for the shape of pullbacks of disks and makes the Epstein class condition superficial. This lemma is formulated below in Appendix 5.2 (Lemma 5.2). Thus, the combination of Lemma 2.4 in [dFdM], the results of [Koz] and of the proof of the renormalization theorem in [LvS] provides Theorem 2.3. The outline of the proof is given in Appendix 5.3.

Theorem 2.1 is a trivial consequence of Theorems 2.2 and 2.3.

3. Polynomial-like maps

The notion of polynomial-like maps was introduced by A. Douady and J. H. Hubbard and was generalized several times after that. The main advantage of using this notion is that one can work with a polynomial-like map in the same way as if it was just a polynomial map. We will use the following definition:

Definition 3.1. A holomorphic map $F : B \to A$ is called *polynomial-like* if it satisfies the following properties:

- B and A are domains in the complex plane, each having finitely many connected components; each connected component of B or A is a simply connected Jordan domain and B is a subset of A. The intersection of the boundaries of the domains A and B is empty or it is a forward invariant set which consists of finitely many points;

- The boundary of a connected component of B is mapped onto the boundary of some connected component of A;
- There is one selected connected component B^c of B (which we will call *central*) such that the map $F|_{B^c}$ is 2-to-1, and the central component B^c is relatively compact in the domain A (i.e. $\bar{B}^c \subset A$);
- On the other connected components of B the map F is univalent.

If the domains A and B are simply connected and the annulus $A \setminus B$ is not degenerate, then a polynomial-like map $F : B \to A$ is called a *quadratic-like* map.

We say that the polynomial-like map is *induced* by the unimodal map f if all connected components of the domains A and B are symmetric with respect to the real line and the restriction of F on the real trace of any connected component of B is an iterate of the map f.

Notice a similarity between polynomial-like maps and holomorphic box maps. There are two differences: in the case of the polynomial-like map the domains A and B consist of several connected components and in the case of the holomorphic box map the domain A is simply connected and the domain B can consist of infinitely many connected components. It is easy to see that if the critical point never leaves B under iterations of F, then the first return map of a polynomial-like map to the connected component of A which contains the critical point is a holomorphic box map.

The main result of this section is that an analytic unimodal map can be "renormalized" to obtain a polynomial-like map.

Before giving the statement of the theorem let us introduce the following notation. $D_\phi(I)$ will denote a lens, i.e. an intersection of two disks of the same radius in such a way that two points of the intersection of the boundaries of these disks are joined by I and the angle of this intersection at these points is 2ϕ. See also Appendix 5.2 and Figure 1.

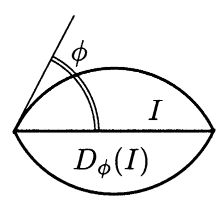

Figure 1. The lens $D_\phi(I)$

THEOREM 3.1. *Let f be an analytic, unimodal, not infinitely renormalizable map with a quadratic recurrent critical point and without neutral periodic points. Then for any $\epsilon > 0$ there exists a polynomial-like map $F : B \to A$ induced by the map f, and satisfying the following properties*:

- *The forward orbit of the critical point under iterations of F is contained in B;*
- *A is a union of finitely many lenses of the form $D_\phi(I)$, where I is an interval on the real line, $|I| < \epsilon$ and $0 < \phi < \pi/4$;*
- *If $F(x) \in A^c$, then B^x is compactly contained in A^x, where B^x and A^x denote connected components of B and A containing x and A^c denotes a connected component of A containing the critical point c (i.e. $\bar{B^x} \subset A^x$, where $\bar{B^x}$ is the closure of B^x);*
- *Boundaries of connected components of B are piecewise smooth curves;*
- *If $a \in \partial A \cap \partial B$, then the boundaries of A and B at a are not smooth; however if we consider a smooth piece of the boundary of A containing a and the corresponding smooth piece of the boundary of B, then these pieces have the second order of tangency (see Figure 2);*
- *If $B^{x_1} \cap B^{x_2} = \emptyset$ and $b \in \partial B^{x_1} \cap \partial B^{x_2}$, then the boundaries of B^{x_1} and B^{x_2} are not smooth at the point b and not tangent to each other;*
- *For any $x \in B$,*
$$\frac{|B^x|}{|A^x|} < \epsilon,$$
where $|B^x|$ denotes the length of the real trace of B^x;
- *If $x \in B$ and $F|_{B^x} = f^n$, then $f^i(x) \notin A^c$ for $i = 1, \ldots, n-1$;*
- *$f(c) \notin A$;*
- *When $a \in \partial A$ is a point closest to the critical value $f(c)$, then*
$$\frac{|f(B^c)|}{|a - f(c)|} < \epsilon.$$

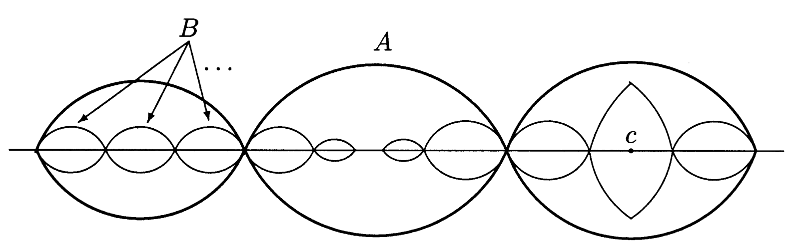

Figure 2. A fragment of the domain of definition of a polynomial-like map

If the map f is infinitely renormalizable, we will use a much simpler statement.

THEOREM 3.2 ([LvS]). *Let f be an analytic unimodal infinitely renormalizable map with a quadratic critical point. Then there exists a quadratic-like map $F : B \to A$ induced by f such that the forward orbit of c under iterates of F is contained in B.*

The proof of Theorem 3.1 will occupy the rest of this section.

3.1. *The real and complex bounds.* In this subsection we give two technical lemmas.

LEMMA 3.1. *Let f be a C^3 nonrenormalizable unimodal map with a quadratic recurrent critical point. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that if T_0 is a sufficiently small nice interval, T_1 is a central domain of T_0, T_2 is a central domain of T_1 and $\frac{|T_1|}{|T_0|} < \delta$, then the following holds: When T_1' is a domain of R_{T_1} containing the critical value $f(c)$ (see Fig. 3), then*

$$\frac{|T_1'|}{|f(T_1)|} < \epsilon.$$

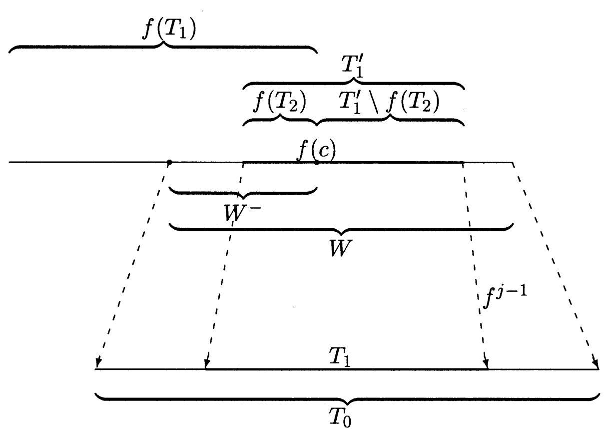

Figure 3. The map f^{j-1}.

$\lhd$ Let $R_{T_1}|_{T_2} = f^j$. The range of the map $f^{j-1} : T_1' \to T_1$ can be extended to the interval T_0 (Lemma 1.1); i.e., there is an interval W such that $f^{j-1} : W \to T_0$ is a diffeomorphism, $T_1' \subset W$ and $f^{j-1}(W) = T_0$. Denote the components of $W \setminus (T_1' \setminus f(T_2))$ as W^- and W^+ in such a way that the interval $f(T_2)$ is a subset of the interval W^-. It is easy to see that the interval

$f(T_1)$ contains the interval W^-. Applying Theorem 1.1 we obtain the following bounds:

$$\begin{aligned}\frac{|T_1'|}{|f(T_1)|} &\leq \frac{|T_1'|}{|W^-|} \leq \mathrm{b}(W, T_1') \\ &\leq \mathrm{b}(T_0, T_1) \leq C_2 \frac{4\delta}{(1+\delta)^2}\end{aligned}$$

where the constant C_2 is close to 1 if the interval T_0 is sufficiently small.

LEMMA 3.2. *Let f be an analytic unimodal map. For any $\phi_0 \in (0, \pi)$ and $K > 0$ there are constants $\phi \in (0, \phi_0)$ and $C_3 > 0$ such that if $f^n|_V$ is monotone, $|f^i(V)| < C_3$ for $i = 0, \dots, n$ and $\sum_{i=0}^{n} |f^i(V)| < K$, then*

$$f^{-n}(D_\phi(f^n(V))) \subset D_{\phi_0}(V).$$

This lemma is a simple consequence of Lemma 5.2 in [dMvS, p. 487]. One can also use Lemma 2.4 in [dFdM] which gives better estimates (see Lemma 5.2).

3.2. *Construction of the induced polynomial-like map.*

Proof of Theorem 3.1. If the ω-limit set of the critical point is minimal (we say that the forward invariant set is *minimal* if it closed and has no proper closed invariant subsets), then one can construct the polynomial-like map in a much simpler way than is given here. In fact, it is a consequence of Theorem 2.3. For example, the domain A in this case is simply connected. However, if the ω-limit set of the critical point contains intervals, the domain A cannot be connected if we want the domain B to contain finitely many connected components.

Letting $\phi_0 = \pi/4$, $K = |X|$, we apply Lemma 3.2 to the map f and obtain two constants ϕ and C_3.

On the other hand, for this constant ϕ there is a constant τ_1 such that if an interval J contains a τ_1-scaled neighborhood of an interval I, then $D_{\pi/4}(I) \subset D_\phi(J)$.

Take a nice interval T_0 such that

- $|T_0| < \epsilon$;
- The boundary points of T_0 are eventually mapped by f onto some repelling periodic point and T_0 is disjoint from the immediate basin of attraction $\mathfrak{B}_0$;
- The central domain T_1 of T_0 is so small that $\frac{|T_1' \setminus f(T_2)|}{|f(T_1)|} < \min(\frac{1}{2}\tan^2\frac{\phi}{2}, \epsilon)$, where T_2 is a central domain of T_1 and T_1' is a domain of R_{T_1} containing the critical value (due to Theorem 2.1 the ratio $\frac{|T_1|}{|T_0|}$ can be made arbitrarily small and then we can apply Lemma 3.1);

- If $f^n|_V$ is monotone and $f^n(V) \subset T_1$, then $|V| < C_3$ (the existence of such an interval T_0 follows from the absence of wandering intervals, for details see Lemma 5.2 in [Koz]);
- Moreover, the ratio $\frac{|T_1|}{|T_0|}$ should be so small that if $f^n|_V$ is monotone and $f^n(V) = T_0$, then V contains a τ_1-scaled neighborhood of the pullback $f^{-n}(T_1)$ and $\frac{|f^{-n}(T_1)|}{|V|} < \epsilon$ (indeed, if $\frac{|T_1|}{|T_0|}$ is small, then the cross-ratio $\mathrm{b}(T_0, T_1)$ is also small, the pullback can only slightly increase this cross-ratio, so that $\mathrm{b}(V, f^{-k}(T_1))$ is small; hence $f^{-k}(T_1)$ is deep inside V).

Let $\mathfrak{B}_0$ be the immediate basin of attraction. It is known that the periods of attracting or neutral periodic points are bounded ([MdMvS]). Hence, the set $X \setminus \bar{\mathfrak{B}}_0$ consists of finitely many intervals (as usual $\bar{\mathfrak{B}}_0$ is a closure of $\mathfrak{B}_0$). Some points of the interval X are mapped to the immediate basin of attraction after some iterates of f. Obviously, for a given n, the set $\{x \in X : f^n(x) \notin \bar{\mathfrak{B}}_0\}$ consists of finitely many intervals as well.

Just to fix the situation let us suppose that the map $f : X \hookleftarrow$ first increases and then decreases. Let $P_n = \{x \in (\partial_- X, f(\partial T_1)) : f^i(x) \notin \bar{T}_1 \cup \bar{\mathfrak{B}}_0 \text{ for } i = 0, \dots, n\}$, where $\partial_- X$ denotes the left boundary point of X. The set P_n consists of finitely many intervals and the lengths of these intervals tend to zero as $n \to \infty$ (otherwise we would have a wandering interval). All the boundary points of P_n are eventually mapped onto some periodic points. Moreover, the set of these periodic points is finite and does not depend on n. Denote the union of this set and $\omega(\partial T_1)$ (which is an orbit of a periodic point by the choice of T_0) by E. Let $a \in E$ be a periodic orbit of period k. Then there exists a neighborhood of a where the map f^k is holomorphically conjugate to a linear map. This implies that if V is a sufficiently small interval and a is its boundary point, then $f^{-2k}(D_\phi(V)) \subset D_\phi(V)$; hence $f^{-2k(i+1)}(D_\phi(V)) \subset f^{-2ki}(D_\phi(V))$ for $i = 0, 1, \dots$ and the size of $f^{-2ki}(D_\phi(V))$ tends to zero.

Due to a theorem of Mãnè there exist two constants $C_4 > 0$ and $\tau_2 > 1$ such that if $x \in P_n$, then $Df^i(x) > C_4 \tau_2^i$ for $i = 0, \dots, n$ (see Theorem 5.1 in [dMvS, p. 248]). Therefore there exists a constant $C_5 > 0$ such that if $V \subset P_n$ is an interval, and $|f^n(V)| < C_5$, then $|f^i(V)| < C_3$ for $i = 0, \dots, n$, and $\sum_{i=0}^n |f^i(V)| < |X|$.

Let m be so large that if V is a connected component of P_m, then $|V| < \min(C_5, \epsilon)$ and, moreover, if V contains a periodic point in its boundary, then V is so small that the lens $D_\phi(V)$ satisfies the properties described above (so it should be in a neighborhood of this periodic point where the map can be linearized and the size of the pullback of $D_\phi(V)$ along this periodic orbit tends to zero).

Once we have fixed the integer m, we are not going to change it and thus we will suppress the dependence of P_m on m.

Let S be a union of the boundary of the set P and the forward orbit of ∂T_1. Notice that S is a finite forward invariant set. The partition of the set $P \cup T_1$ by points of S we denote by $\mathcal{P}$. Finally, let $A = \bigcup_{V \in \mathcal{P}} D_\phi(V)$. The set A will be the range of the polynomial-like map we are constructing.

Let Σ be a closure of all points on the real line whose ω-limit set contains the critical point. For any point $x \in \Sigma' = \Sigma \cap (\bar{P} \cup \bar{T}_1)$ such that $f^i(x) \notin E$ for any $i > 0$, we will construct an interval $I(x)$ and an integer $n(x)$ such that $x \in I(x)$, $f^{n(x)}(I(x)) \in \mathcal{P}$ and $f^{-n(x)}(D_\phi(\mathcal{P}(f^{n(x)}(x)))) \in D_\phi(\mathcal{P}(x))$, where $\mathcal{P}(x)$ denotes an element of the partition containing the point x. If the point $x \in \Sigma'$ is eventually mapped to some point of E and on both sides of x there are points of Σ' arbitrarily close to x, then we will construct two intervals $I_-(x)$ and $I_+(x)$ on both sides of x and two integers $n_-(x)$ and $n_+(x)$ with similar properties. If $f^i(x) \in E$ but there are no points of Σ' on one side of x close to x, only intervals on the side containing points of Σ' will be constructed. Finally, if $x \in T_2$, we will put $I(x) = T_2$ and $n(x)$ will be a minimal positive integer such that $f^{n(x)}(x) \in T_1$. In this case $f^{n(x)}(I(x)) \subsetneq T_1$ and so $f^{n(x)}(I(x)) \notin \mathcal{P}$, however as we will see below $f^{-n(x)}(D_\phi(T_1)) \subset D_\phi(T_1)$.

First, we are going to construct these intervals and integers for a point x whose orbit contains points of the set S, where S is a set of boundary points of P. In this case some iterate of x lands on a periodic point $a \in E$; i.e., $f^k(x) = a \in E$. For simplicity let us assume that a is just a fixed point and that its multiplier is positive. Let J be an interval of $\mathcal{P}$ containing a (there are at most two such intervals). Because of the choice of m we know that $f|_J^{-1}(D_\phi(J)) \subset D_\phi(J)$ and since $D_\phi(J)$ is in the neighborhood of a where the map f can be linearized, the sizes of domains $f|_J^{-i}(D_\phi(J))$ shrink to zero when $i \to +\infty$. Thus, there exists i_0 such that

$$f^{-k} \circ f|_J^{-i_0}(D_\phi(J)) \subset D_\phi(J')$$

and

$$\frac{|f^{-k} \circ f|_J^{-i_0}(J)|}{|J'|} < \epsilon,$$

where J' is just $\mathcal{P}(x)$ if $x \notin S$ and J' is one of the intervals of $\mathcal{P}$ which contains x on its boundary if $x \in S$. We put $I_-(x) = f^{-k} \circ f|_J^{-i_0}(J)$ and $n_-(x) = k + i_0$. If there is another interval from $\mathcal{P}$ containing a in its boundary, we can repeat the procedure and get the interval $I_+(x)$ and the integer $n_+(x)$; otherwise we are finished in this case.

Now let us consider the case when $f^i(x) \notin S$ for all $i > 0$. This case we divide in several subcases.

If $x \in T_2$, then $I(x) = T_2$ and $n(x)$ is a minimal positive integer such that $f^{n(x)}(T_2) \subset T_1$; i.e., $R_{T_1}|_{T_2} = f^{n(x)}$. Let T_1' be an interval around the critical value $f(c)$ such that $f^{n(x)-1}(T_1') = T_1$ (see Figure 3). The pullback of

a lens $D_\phi(T_1)$ by $f^{-(n(x)-1)}$ is contained in $D_{\pi/4}(T_1')$ (indeed, by the choice of T_0 we know that all intervals in the orbit $\{f^i(T_1'),\ i=0,\dots,n(x)\}$ are small and they are disjoint; so we can apply Lemma 3.2). Near the critical point the map f is almost quadratic (if T_0 is small enough) and because of the choice of T_0 the interval $f(T_1)$ is much larger than the part of the interval T_1' which is on the other side of the critical value. Therefore, the pullback $f^{-n(x)}(D_\phi(T_1))$ is contained in the lens $D_\phi(T_1)$.

Another subcase is the following: suppose that $f^k(x) \in T_1$ ($x \in (P \cup T_1) \setminus T_2$) and let k be a minimal positive integer satisfying this property. Put $I(x) = f^{-k}(T_1)$ and $n(x) = k$. Due to Lemma 1.1 the range of the map $f^k|_{I(x)}$ can be extended to T_0. The pullback of T_0 by f^{-k} along the orbit of x which we denote by W, is contained in $\mathcal{P}(x)$. Indeed, suppose that $W \cap S$ is nonempty, so that there is a point $y \in W \cap S$, and consider two cases. If $x \in T_1$, then $y \in \partial T_1$ and we would have $f^k(y) \in T_0$ which contradicts the fact that iterates of the boundary points of T_1 never return to the interior of T_0. On the other hand, if $x \in P$, then $k > m$ because otherwise we would have $x \notin P$. Now, $f^m(y)$ is either a periodic point belonging to the boundary of $\mathfrak{B}_0$ or a point of the forward orbit of the boundary of T_1; thus in any case the point $f^k(y)$ cannot be inside of T_0. In both cases we have obtained contradictions, therefore $W \subset \mathcal{P}(x)$.

By the choice of T_0 we know that W contains a τ_1-scaled neighborhood of $I(x)$, the intervals in the orbit of $\{f^i(I(x)),\ i=0,\dots k-1\}$ are small and since $I(x)$ is a domain of the first entry map to T_1 the orbit is disjoint. Hence we can see that $f^{-k}(D_\phi(T_1)) \subset D_{\pi/4}(I(x)) \subset D_\phi(\mathcal{P}(x))$ (see the choice of the constant τ_1 in the beginning of the proof).

The last case to consider is the case when $f^i(x) \notin T_1$ for all $i > 0$. Then $f^i(x) \in \bar{P}$ for all $i > 0$. Indeed, if $f^i(x) \notin \bar{P}$ for some i, then either $f^i(x) \in [f(\partial T_1), \partial_+ X]$ or $f^{i+j}(x) \in \bar{\mathfrak{B}}_0$ for some $j \le m$. In the former case we would have $f^{i-1}(x) \in T_1$ (contradiction) and the latter case is impossible because any point of Σ avoids $\mathfrak{B}_0$. Thus, x belongs to the hyperbolic set described above, and the sizes of intervals $f^{-i}(\mathcal{P}(f^i(x)))$ go to zero as $i \to \infty$. Take k to be so large that $\mathcal{P}(x)$ is a τ_1-scaled neighborhood of $f^{-k}(\mathcal{P}(f^k(x)))$ and

$$\frac{\left|f^{-k}(\mathcal{P}(f^k(x)))\right|}{|\mathcal{P}(x)|} < \epsilon.$$

Put $n(x) = k$ and $I(x) = f^{-k}(\mathcal{P}(f^k(x)))$. By the choice of m we know that $|\mathcal{P}(f^k(x))| < C_5$, hence $|f^i(I(x))| < C_3$ for $i = 0,\dots,k$ and $\sum_{i=0}^k |f^i(I(x))| < |X|$. As in the previous case we have $f^{-k}(D_\phi(\mathcal{P}(f^k(x)))) \subset D_{\pi/4}(I(x)) \subset D_\phi(\mathcal{P}(c))$.

So, we have assigned to each point of Σ' one or two intervals. Now we will show that there are finitely many intervals of this form whose closures cover all points in Σ'. First we will slightly modify these intervals.

When $x \in \Sigma'$, we have assigned to it just one interval which contains x in its interior. Then we let $\mathring{I}(x)$ be the interior of $I(x)$. Another case: we have assigned to x one interval, say, $I_-(x)$, but x is its boundary point. Then on the other side of x there is a point y such that the interval (x, y) does not contain points from the set Σ'. In this case $\mathring{I}(x)$ is a union of the interior of $I_-(x)$ and the half interval $[x, y)$. The last case: there are two intervals assigned to x. Let $\mathring{I}(x)$ be the interior of $I_-(x) \cup I_+(x)$.

We have covered all points in Σ' by open intervals. The set Σ' is compact, therefore there exist finitely many such intervals which cover Σ'. Let us denote these intervals by $\mathring{I}(x_1), \mathring{I}(x_2), \dots, \mathring{I}(x_N)$. Now, instead of these intervals consider all the intervals which are assigned to the points $x_1, \dots, x_N$, i.e. intervals of the form $I_p(x_i)$, where p is either void or $-$ or $+$ and $i = 1, \dots, N$. Obviously, the closures of these closed intervals also cover Σ'. Moreover, it is easy to see that if the interiors of two intervals from this set intersect, then one of them is contained in the other. This is a consequence of the fact that the set S is forward invariant and the boundary points of $I(x)$ are eventually mapped into S. Thus, there exists a finite collection of intervals of the form $I(x)$ ($I_\pm(x)$) such that the closures of these intervals cover the whole set Σ' and these intervals can intersect each other only in the boundary points. Denote this intervals by $I_1, \dots, I_k$.

By the construction for each interval I_i there is an integer n_i associated to it. Let $B_i = f^{-n_i}(D_\phi(\mathcal{P}(f^{n_i}(I_i))))$. We have the following properties of I_i, n_i and B_i:

- $f^{n_i}(I_i) \in \mathcal{P}$ and $f^{n_i}|_{I_i}$ is monotone if $I_i \neq T_2$;
- If $I_i = T_2$, then $f^{n_i}|_{I_i}$ is unimodal;
- If $I_i \subset J \in \mathcal{P}$, then $B_i \subset D_\phi(J)$;
- If $I_i \neq T_2$, then $B_i \subset D_{\pi/4}(I_i)$, thus the domains B_i are disjoint.

Let $B = \cup_{i=1}^k B_i$. It follows that B is a subset of A. If $x \in B_i$, put $F(x) = f^{n_i}(x)$.

By the very construction of F one can see that it satisfies all the required properties. $\square$

4. C^ω structural stability

Here we will prove the C^k structural stability conjecture.

THEOREM A. *Axiom* A *maps are dense in the space of* $C^\omega(\Delta)$ *unimodal maps in the* $C^\omega(\Delta)$ *topology* (Δ *is an arbitrary positive number*).

We define $C^\omega(\Delta)$ to be the space of real analytic functions defined on the interval which can be holomorphically extended to a Δ-neighborhood of this interval in the complex plane.

Let us recall that the map f is *regular* if either the ω-limit set of the critical point does not contain neutral periodic points or the ω-limit set of c coincides with the orbit of some neutral periodic point. Any map having negative Schwarzian derivative is regular. In Section 4.5 we will see that any analytic map f without neutral periodic points can be included in the family of regular analytic maps.

THEOREM C. *Let* $f_\lambda : X \hookleftarrow$ *be an analytic family of analytic unimodal regular maps with a nondegenerate critical point*, $\lambda \in \Omega \subset \mathbb{R}^N$ *where* Ω *is an open set. If the family* f_λ *is nontrivial in the sense that there exist two maps in this family which are not combinatorially equivalent*, *then Axiom* A *maps are dense in this family. Moreover*, *let* Υ_{λ_0} *be a subset of* Ω *such that the maps* f_{λ_0} *and* $f_{\lambda'}$ *are combinatorially equivalent for* $\lambda' \in \Upsilon_{\lambda_0}$ *and the iterates of the critical point of* f_{λ_0} *do not converge to some periodic attractor. Then the set* Υ_{λ_0} *is an analytic variety. If* $N = 1$, *then* $\Upsilon_{\lambda_0} \cap Y$, *where the closure of the interval* Y *is contained in* Ω, *has finitely many connected components.*

Remark. In Section 4.1 it will be shown that the regularity condition is superficial if one is concerned only about infinitely renormalizable maps (or more generally, maps whose ω-limit set of the critical point is minimal). Thus, the following statements holds: Let $f_\lambda : X \hookleftarrow$ be an analytic nontrivial family of analytic unimodal maps with a nondegenerate critical point, $\lambda \in \Omega \subset \mathbb{R}$, where Ω is an open set. If the ω-limit set of the critical point of the map f_{λ_0} is minimal, then the set $\Upsilon_{\lambda_0} \cap Y$, where the closure of the interval Y is contained in Ω, consists of finitely many points.

In order to underline the main idea of the proof of this theorem we split it into three parts. First we assume that the map f is infinitely renormalizable. In this case the induced quadratic-like map is simpler to study than the induced polynomial-like map in the other case. After proving the theorem in this case we will explain why some extra difficulties in the general case emerge and then we will show how to overcome them. Finely we consider the case of Misiurewicz maps (which is the simplest case).

For the reader's convenience we collect all theorems about quasi-conformal maps which we will use intensively in Appendix 5.

4.1. *The case of an infinitely renormalizable map.* In this section we will proof the following lemma:

LEMMA 4.1. *Let $f_\lambda : X \hookleftarrow$ be an analytic family of analytic unimodal maps with a nondegenerate critical point, $\lambda \in \Omega \subset \mathbb{R}^N$ where Ω is a open set. Suppose that the map f_{λ_0} is infinitely renormalizable. Then there is a neighborhood Ω' of λ_0 such that the set $\Upsilon_{\lambda_0} \cap \Omega'$ is an analytic variety.*

This lemma remains true if instead of assuming that the map f_{λ_0} is infinitely renormalizable, we assume that the ω-limit set of the critical point of this map is minimal. Note that we do not assume here that the family f is regular.

We can assume that $\lambda_0 = 0$.

From Theorem 3.2 we know that if the map is analytic and infinitely renormalizable, then there is an induced quadratic-like map $F_0 : B \to A$, where $B \subset A \subset \mathbb{C}$ are simply connected domains and the modulus of the annulus $A \setminus B$ is not zero.

The map F_0 is the extension of some iterate of the map f_0 to the domain B, i.e., $F_0|_B = f_0^n$. If we take a small neighborhood $D \subset \mathbb{C}^N$ of 0 in the parameter space, then the map $F_\lambda = f_\lambda^n$ will have the extension to some domain which contains B for any $\lambda \in D$. Fix the domain A and let B_λ be a preimage of the domain A under the map F_λ where $\lambda \in D$ and let $B_\lambda \subset A$.

Define the map $\phi_\lambda : \partial B_0 \cup \partial A \to \partial B_\lambda \cup \partial A$ by the following formula: $\phi_\lambda(z) = F_\lambda^{-1} \circ F_0(z)$ where $\lambda \in D$, $z \in \partial B_0$ and $\phi_\lambda(z) = z$ for $z \in \partial A$. The map F_λ is not invertible, but if ϕ is continuous with respect to λ and $\phi_0 = \mathrm{id}$, then it is defined uniquely.

For fixed z the map $\phi_\lambda(z)$ is holomorphic with respect to λ. Shrinking the neighborhood D if necessary, we can suppose that the map $z \mapsto \phi_\lambda(z)$ is injective for fixed $\lambda \in D$. Due to λ-lemma (Theorem 5.3) the map ϕ_λ can be extended to the annulus $A \setminus B_0$ in the q.c. (quasiconformal) way. Denote this extension by $h_\lambda^0 : A \setminus B_0 \to A \setminus B_\lambda$. Thus, h_λ^0 is a q.c. homeomorphism and its Beltrami coefficient ν_λ^0 is a holomorphic function with respect to $\lambda \in D$.

Denote the pullback of the Beltrami coefficient ν_λ^0 by the map F_0 as ν_λ; i.e., if $F_0^k(z) \in A \setminus B$, then $\nu_\lambda(z) = F_0^{k\,*}\nu_\lambda^0(F_0^k(z))$. On the filled Julia set of F_0 and outside of the domain A we set ν_λ equal to 0. It is easy to see that since $\lambda \mapsto \nu_\lambda^0(z)$ is analytic the map $\lambda \mapsto \nu_\lambda(z)$ is analytic as well.

According to the measurable Riemann mapping Theorem 5.1 below, there is a family q.c. homeomorphism $h_\lambda : \mathbb{C} \to \mathbb{C}$ whose Beltrami coefficient is ν_λ and which is normalized such that $h_\lambda(\infty) = \infty$, $h_\lambda(a^-) = a^-$, $h_\lambda(a^+) = a^+$ where the $a^\pm$ are two points of the intersection of ∂A and the real line.

Since the map F_0 conserves the Beltrami coefficient ν_λ the map

$$G_\lambda = h_\lambda \circ F_0 \circ h_\lambda^{-1} : B_\lambda \to A$$

is holomorphic. Due to the Ahlfors-Bers Theorem 5.2 the map $\lambda \mapsto G_\lambda(z)$ is analytic for the fixed point z. Thus G is an analytic family of holomorphic quadratic-like maps.

LEMMA 4.2. *The maps f_0 and f_λ are combinatorially equivalent if and only if $F_\lambda = G_\lambda$.*

$\lhd$ If $F_\lambda = G_\lambda$, then F_λ and F_0 are topologically conjugate; hence f_λ and f_0 are combinatorially equivalent.

If f_0 and f_λ are combinatorially equivalent, then the maps F_0 and F_λ are combinatorially equivalent as well. Due to the rigidity theorem and straightening Theorem 5.7 we know that there is a q.c. homeomorphism $\tilde{H} : \mathbb{C} \to \mathbb{C}$ which is a conjugacy between F_0 and F_λ on their Julia sets; i.e., $\tilde{H} \circ F_0|_J = F_\lambda \circ \tilde{H}|_J$ where J is the Julia set of the map F_0.

Define a new q.c. homeomorphism H^0 in the following way:

$$H^0(z) = \begin{cases} z & \text{if } z \notin A \\ h_\lambda^0(z) & \text{if } z \in A \setminus B \\ \tilde{H}(z) & \text{if } z \in B(J) \end{cases}$$

where $B(J)$ is a neighborhood of the Julia set J such that $B(J) \subset B$. In the annulus $B \setminus B(J)$ the q.c. homeomorphism H^0 is defined in an arbitrary way.

Consider the sequence of q.c. homeomorphisms H^i which are defined by the formula $H^{i+1} = F_\lambda^{-1} \circ H^i \circ F_0$. The map F_λ is not invertible, but H^{i+1} is defined correctly because of the homeomorphism $\tilde{H}$ and as a consequence the homeomorphism H^i maps the orbit of the critical point of F_0 onto the orbit of the critical point of F_λ. Since the maps F_0 and F_λ are holomorphic the distortion of H^i does not increase with i. So the sequence $\{H^i\}$ is normal and we can take a subsequence convergent to some limit $\hat{H}$ which is also a q.c. homeomorphism. Taking a limit in the equality $H^{i+1} = F_\lambda^{-1} \circ H^i \circ F_0$ we obtain that the homeomorphism $\hat{H}$ is a conjugacy between F_0 and F_λ; i.e., $F_\lambda \circ \hat{H} = \hat{H} \circ F_0$. On the other hand, it is easy to see that the Beltrami coefficient of $\hat{H}$ coincides with the Beltrami coefficient ν_λ. Indeed, outside of A both coefficients are zero. In the domain $A \setminus J$ both coefficients are obtained by pulling back the Beltrami coefficient ν_λ^0. On the Julia set the Beltrami coefficient of $\hat{H}$ is equal to the Beltrami coefficient of $\tilde{H}$ which is 0 because of the rigidity theorem. The homeomorphism $\hat{H}$ is normalized in the same way as h_λ, so that by the measurable Riemann mapping theorem these homeomorphisms coincide. From the very definition of the map G_λ we obtain that $F_\lambda = G_\lambda$. $\rhd$

Due to the previous lemma f_0 and f_λ are combinatorially equivalent if and only if $F_\lambda = G_\lambda$. So, the solution with respect to λ of the equation $F_\lambda = G_\lambda$ is the set $\Upsilon_0 \cap D$. Since this equation is holomorphic, its solution is an analytic variety.

4.2. *The case of a finitely renormalizable* (*nonrenormalizable*) *map.* In the previous section the domain $A \setminus B$ had the nice boundary which was a union of two Jordan curves. In the general case this is false. Indeed, recall the structure of the domains A and B which is given in Section 3. The domain A is a union of finitely many lenses based on the real line. Inside of each lens there are finitely many quasilenses which are connected components of the domain B (see Figure 2). Thus, if $A^{x_0} \subset A$ is a connected component of the domain A, then the set $A^{x_0} \setminus B$ consists of 1 or 2 connected components which can have cusps or angles on their boundaries (recall that A^x denotes a connected component of A containing the point x).

Notice that the family f_λ consists of regular maps so that we will not have neutral periodic points on the boundary of the domains A and B.

Let a be a periodic point from the set $E = \omega(\partial(A \cap \mathbb{R}))$ (see §3.2). For simplicity we will assume that the point a is a fixed point. Denote the multiplier of the map F_λ at the point a as d_λ and let ∂A^{x_0} and ∂B^{x_0} contain the point a. If on the boundary of the domain A^{x_0} we define the map h^0_λ to be the identity, then on the boundary of the domain B near the point a we will have $h^0_\lambda(z) = d_0/d_\lambda\, z + \cdots$ because the map h^0_λ has to conjugate the maps F_0 and F_λ on the boundary of B; i.e., $h^0_\lambda|_{\partial A} \circ F_0|_{\partial B} = F_\lambda|_{\partial B_\lambda} \circ h^0_\lambda|_{\partial B}$. At the point a the boundaries of the domains B and A are tangent to each other, and if the multiplier d_λ changes with λ, then the derivative of h^0_λ in the direction of ∂A is 1 and in the direction of ∂B is d_0/d_λ. One can easily check that a homeomorphism h^0_λ defined on the domain $A \setminus B$ cannot be quasiconformal.

As a result of this discussion we conclude that we have to deform the domain A_λ as well in order to construct the q.c. homeomorphism h^0_λ.

Now we will prove Lemma 4.1 in the case when the map f_0 is finitely renormalizable.

LEMMA 4.3. *Let $f_\lambda : X \hookleftarrow$ be an analytic regular family of analytic unimodal maps with a nondegenerate critical point, $\lambda \in \Omega \subset \mathbb{R}^N$ where Ω is an open set. Suppose that the map f_{λ_0} is finitely renormalizable. Then there is a neighborhood Ω' of λ_0 such that the set $\Upsilon_{\lambda_0} \cap \Omega'$ is an analytic variety.*

Recall the notation used in Section 3.2. According to Theorem 3.1, for our map f_0 there is an induced polynomial-like map $F_0 : B_0 \to A_0$. The set S consists of points where the domain A_0 has singularities. This set is finite and forward invariant, so that it has periodic points and let E denote this subset

of periodic points. Any point from the set S is mapped into E after some iterations.

We can make an analytic change of the coordinate which also depends on the parameter λ analytically in such a way that the set S does not move with the parameter λ for small λ. So in this section we will assume that the set S does not depend on λ.

Take any periodic point r from the set E and let m be the period of this periodic point r. Let x be a local coordinate in the neighborhood of the point r and let the map f_λ^m have the following series expansion:

$$f_\lambda^m(x) = d_\lambda x + q_\lambda x^2 + O(x^3).$$

The coefficients d_λ and q_λ depend analytically on the parameter λ.

Our goal is the construction of a q.c. homeomorphism $h_\lambda^0 : A_0 \setminus B_0 \to \mathbb{C}$ which conjugates the maps F_0 and F_λ on the domain $A_0 \setminus B_0$.

Assume that $d_\lambda > 0$ and let $A_0^{x_0} \supset B_0^{x_0}$ be connected components of the domains A_0 and B_0 which have the point r in their boundaries. It follows from the construction of the domains A_0 and B_0 that at the point r the boundaries of A^{x_0} and B^{x_0} are tangent to each other and that this tangency is quadratic. We will look for the map h_λ^0 near the point r in the following form:

$$h_\lambda^0(z) = (z-r)^{l_\lambda} b_\lambda(z-r)(1+o(z-r)),$$

where $b(z)$ is a holomorphic function such that $b(0) \neq 0$.

Since the map h_λ^0 should conjugate the maps F_0 and F_λ we obtain the following equation for h_λ^0:

$$h_\lambda^0 \circ f_0^m = f_\lambda^m \circ h_\lambda^0.$$

Solving this equation we obtain the series expansion of h_λ^0:

$$h_\lambda^0(z) = (z-r)^{l_\lambda} + \alpha_\lambda (z-r)^{2l_\lambda} + \beta_\lambda (z-r)^{l_\lambda+1} + O((z-r)^\kappa)$$

where

$$\begin{aligned} l_\lambda &= \frac{\ln(d_\lambda)}{\ln(d_0)}, \qquad \alpha_\lambda = \frac{q_\lambda}{d_\lambda^2} \\ \beta_\lambda &= \frac{l_\lambda q_0}{d_0(1-d_0)}, \qquad \kappa = \min(3l_\lambda, 2l_\lambda+1). \end{aligned}$$

Now to each point of the set S we associate a jet by the following rule: first, from each periodic orbit of the set E take a representative and denote this set of representatives as E'. For a point $r \in E'$ the corresponding jet $j_{r,\lambda}$ is defined as $x^{l_\lambda} + \alpha_\lambda x^{2l_\lambda} + \beta_\lambda x^{l_\lambda+1} + O(x^\kappa)$ where l_λ, α_λ and β_λ are calculated according to the formulas above. If $a \in S \setminus E'$, then some iteration of a is mapped into the set E', so that $f^n(a) = r$ where r is some element of the set E'. Then at the point a the jet $j_{a,\lambda}$ is defined as $f_\lambda^{-n} \circ j_{r,\lambda} \circ f_0^n$. Certainly, we truncate the terms of order $O(x^\kappa)$ and higher.

Now, at each point of the set S we have a jet which depends on the parameter λ.

The family of maps $\phi_\lambda : \partial A_0 \cup \partial B_0 \to \mathbb{C}$ will be defined first on the boundary of the domain A_0. Let it satisfy the following conditions:

- $\phi_0 = \mathrm{id}$;
- For fixed $z \in \partial A$ the map $\lambda \mapsto \phi_\lambda(z)$ is analytic;
- For fixed λ the map $z \mapsto \phi_\lambda(z)$ is differentiable and nonneutral for $z \in \partial A_0 \setminus S$;
- For any $r \in S$ we have $\phi_\lambda(z) = j_{r,\lambda}(z-r) + O((z-r)^\kappa)$.

One can easily construct the map ϕ_λ satisfying these conditions.

On the boundary of the domain B_0 we define the map ϕ_λ in such a way that ϕ_λ conjugates the maps F_0 and F_λ; i.e.,

$$\phi_\lambda|_{\partial A} \circ F_0|_{\partial B} = F_\lambda|_{\partial B_\lambda} \circ \phi_\lambda|_{\partial B}.$$

Thus

$$\phi_\lambda|_{\partial B_0} = F_\lambda^{-1}|_{\partial A_\lambda} \circ \phi_\lambda|_{\partial A_0} \circ F_0|_{\partial B_0}$$

where $\partial A_\lambda = \phi_\lambda(\partial A_0)$.

From the construction it follows that at the points where the domain $A_0 \setminus B_0$ has quadratic singularities (i.e. at points of the set S) we have

$$\phi_\lambda(z-a) = \gamma_{a,\lambda}(z-a)^{l_\lambda} + \alpha_{a,\lambda}(z-a)^{2l_\lambda} + \beta_{a,\lambda}(z-a)^{l_\lambda+1} + O((z-a)^\kappa)$$

where $a \in S$ and $z \in \partial A_0 \cup \partial B_0$.

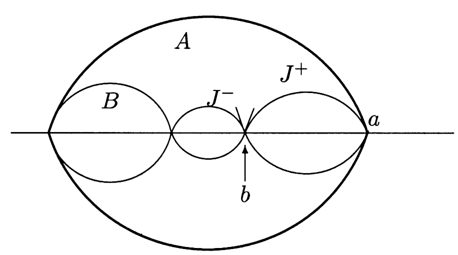

Figure 4. A connected component of the domain A_0. At the point b the angle is not zero.

If b is a singularity of the domain $A \setminus B$ where this domain has a nonzero angle (i.e. b is a point of the intersection of the closure of two connected components of the domain B_0), denote two arcs which are boundary arcs of the domain B and which intersect at b, as J^- and J^+ (see Fig. 4). Let $F_0|_{J^i} = f^{k_i}$ for $i = -, +$. The numbers k_- and k_+ do not necessarily coincide. Therefore, the jets of the maps $\phi_\lambda|_{J^-}$ and $\phi_\lambda|_{J^+}$ are different. However, the exponents of the leading terms of these jets do coincide. So, in the neighborhood of the point b we have

$$\phi_\lambda(z) = \gamma_{i,\lambda}(z-b)^{l_\lambda}(1 + O((z-b)^{\min(l_\lambda,1)}))$$

for $z \in J^i$ where $i = -, +$, and $\gamma_{i,\lambda}$ is holomorphic with respect to λ, $\gamma_{i,0} \neq 0$ and $\gamma_{i,\lambda}$ is real for real λ.

LEMMA 4.4. *There is a small neighborhood $D \subset \mathbb{C}^N$ of 0 such that for fixed $\lambda \in D$ the map $\phi_\lambda : A_0 \setminus B_0 \to \mathbb{C}$ defined above is injective.*

$\lhd$ First, we will check that the map ϕ_λ is injective in some small neighborhood of the point b where we have a nonzero angle.

Let x be a local coordinate in the neighborhood of b and let the curves J^- and J^+ have the parametrizations $x = u_- t + O(t^2)$ and $x = u_+ t + O(t^2)$, where $t \in \mathbb{R}$ and $u_-, u_+ \in \mathbb{C}$. Since the angle at b is nonzero the ratio $\frac{u_-}{u_+}$ cannot be real.

Suppose that ϕ_λ is not injective. Then there are real numbers t_- and t_+ such that

$$\gamma_{-,\lambda}\, u_-^{l_\lambda}\, t_-^{l_\lambda}\,(1 + O(t_-^{\min(l_\lambda,1)})) = \gamma_{+,\lambda}\, u_+^{l_\lambda}\, t_+^{l_\lambda}\,(1 + O(t_+^{\min(l_\lambda,1)})).$$

For small λ the exponent l_λ is close to 1. Hence, for small λ the imagery part of $\frac{\gamma_{-,\lambda}}{\gamma_{+,\lambda}}\left(\frac{u_-}{u_+}\right)^{l_\lambda}$ is bounded away from 0. Thus, for small λ and t_-, t_+ the equation

$$\frac{\gamma_{-,\lambda}}{\gamma_{+,\lambda}}\left(\frac{u_-}{u_+}\right)^{l_\lambda} = \left(\frac{t_-}{t_+}\right)^{l_\lambda}(1 + O(t_-^{\min(l_\lambda,1)}) + O(t_+^{\min(l_\lambda,1)}))$$

does not have real solutions.

Consider now the point $a \in S$ where we have a quadratic singularity. Let us again parametrize the boundaries of A_0 and B_0 in the neighborhood of a by $x = ut + v_- t^2 + O(t^3)$ and $x = ut + v_+ t^2 + O(t^3)$ where u is a complex number, v_-, v_+ are real numbers and $v_- \neq v_+$.

The equation we have to solve is the following:

$$\begin{aligned}\gamma_\lambda\,(ut_- + v_- t_-^2)^{l_\lambda} + \alpha_\lambda\,(ut_- + v_+ t_-^2)^{2l_\lambda} + \beta_\lambda\,(ut_- + v_- t_-^2)^{l_\lambda+1} + O(t_-^{\kappa_\lambda})\\ = \gamma_\lambda\,(ut_+ + v_- t_+^2)^{l_\lambda} + \alpha_\lambda\,(ut_+ + v_+ t_+^2)^{2l_\lambda} + \beta_\lambda\,(ut_+ + v_- t_+^2)^{l_\lambda+1} + O(t_+^{\kappa_\lambda}).\end{aligned}$$

After simplification we obtain:

$$\gamma_\lambda(ut_-)^{l_\lambda} + \gamma_\lambda l_\lambda u^{l_\lambda-1} v_- t_-^{l_\lambda+1} + \alpha_\lambda(ut_-)^{2l_\lambda} + \beta_\lambda(ut_-)^{l_\lambda+1} + O(t_-^{\kappa_\lambda})$$
$$= \gamma_\lambda(ut_+)^{l_\lambda} + \gamma_\lambda l_\lambda u^{l_\lambda-1} v_+ t_+^{l_\lambda+1} + \alpha_\lambda(ut_+)^{2l_\lambda} + \beta_\lambda(ut_+)^{l_\lambda+1} + O(t_+^{\kappa_\lambda}).$$

One can easily see that this equality implies that $t_- = t_+ + \frac{v_+ - v_-}{u} t_+^2 + o(t_+^2)$. However, $v_+ - v_-$ is a real number and u is complex, so if t_+ is a small real number, then t_- is complex. Thus, for small λ the map ϕ_λ is injective in small neighborhoods of the singular points.

If at some point the boundary of B_0 or A_0 is smooth, then for small λ the map ϕ_λ is injective as well in some neighborhood of this point. By compactness arguments we obtain that for small λ the map ϕ_λ is injective. $\triangleright$

According to the λ-lemma we can extend the map ϕ_λ to the domain $A_0 \setminus B_0$. In other words, there is a family of q.c. homeomorphisms $h_\lambda^0 : A_0 \setminus B_0 \to \mathbb{C}$ where λ is in some small neighborhood of the point 0. This family satisfies the following conditions:

- $h_0^0 = \mathrm{id}$;
- For the fixed parameter λ the map h_λ^0 is a q.c. homeomorphism and $h_\lambda^0|_{\partial A \cup \partial B} = \phi_\lambda$;
- For fixed $z \in A_0 \setminus B_0$ the maps $\lambda \mapsto h_\lambda^0(z)$ and $\lambda \mapsto \nu_\lambda^0(z)$ are analytic where ν_λ^0 is the Beltrami coefficient of h_λ^0.

Now we have the map h_λ^0, so we can construct the q.c. homeomorphism h_λ and the analytic family G. Lemma 4.2 still holds, but we have to alter its proof because we cannot use the straightening theorem any more. Instead of it we will use the following theorem (see [GS], [Lyu4]).

THEOREM 4.1. *Let $R_0 : \hat{B}_0 \to \hat{A}_0$ and $R_1 : \hat{B}_1 \to \hat{A}_1$ be holomorphic box mappings such that R_0 and R_1 are combinatorially equivalent, and the moduli of the annuli $\hat{A}_i \setminus \hat{B}_i^x$ are uniformly bounded away from zero for all $x \in \hat{B}_i \cap \mathbb{R}$, where B_i^x is a connected component of $\hat{B}_i$ containing the point x, $i = 0, 1$. Moreover, suppose that there is a quasisymmetric homeomorphism Q such that $Q \circ R_0|_{\partial \hat{B}_0 \cap \mathbb{R}} = R_1 \circ Q|_{\partial \hat{B}_1 \cap \mathbb{R}}$. Then the maps R_0 and R_1 are* q.c. *conjugate on their postcritical sets.*

Consider the map $F_0 : B_0 \to A_0$ which is induced by the map f_0. Let A_0^c be a connected component of A_0 which contains the critical point. If B_0^x is a connected component of B_0 which is mapped onto A_0^c by F_0 (this is equivalent to saying that $F_0(x) \in A_0^c$), then the domain B_0^x is disjoint from the boundary of the domain A_0 (see Theorem 3.1). Since there are only finitely many connected components of the domain B_0 we see that there is a positive number C_6 such that $\mathrm{mod}(A_0^x \setminus B_0^x) > C_6$ for any $x \in B_0 \cap \mathbb{R}$ such that $F_0(x) \in A_0^c$.

Denote the first return map of the map F_0 to the domain A_0^c by R_0 and A_0^c by $\hat{A}_0$. It is easy to see that R_0 is a holomorphic box mapping and that the moduli of the annuli $\hat{A}_0 \setminus \hat{B}_0^x$ with $x \in \hat{B}_0 \cap \mathbb{R}$ are uniformly bounded away from zero by the constant C_6, where $\hat{B}_0^x \subset \hat{A}_0$ is a connected component of the domain of definition of the map R_0.

In a similar way we can define the first entry map R_λ. In order to apply the previous theorem to the maps R_0 and R_λ and to find the q.c. conjugacy between R_0 and R_λ on their postcritical sets we have to construct the q.s. homeomorphism Q. It is easy to do using the following observations: first, the maps f_0 and f_λ are regular, hence they have no neutral periodic points (we have supposed that the critical points are recurrent); in this case the set of points which do not belong to the basin of attraction and whose iterates do not enter some neighborhood of the critical point is a hyperbolic set; since the maps f_0 and f_λ are conjugate these corresponding hyperbolic sets are conjugate as well and this conjugacy Q is quasi-symmetric. This can be proved using the same ideas as for the Misiurewicz maps; see, for example, [dMvS]. Obviously, the set $\partial\hat{B}_0 \cap \mathbb{R}$ is a subset of the hyperbolic set which consists of points whose iterates do not enter the interval $\hat{B}_0^c \cap \mathbb{R}$ (and do not belong to the basin of attraction). Another way to see the existence of this q.s. homeomorphism in our case is the following: the set $\partial\hat{B}_\lambda \cap \mathbb{R}$ consists of preimages of points in $\partial B_\lambda \cap \mathbb{R}$ and it varies holomorphically with respect to λ. Moreover, this set is a part of some hyperbolic set, hence it persists for small $|\lambda|$ (even if λ is complex). Now we can apply the λ-lemma and get a q.c. homeomorphism which maps $\partial\hat{B}_0 \cap \mathbb{R}$ onto $\partial\hat{B}_\lambda \cap \mathbb{R}$.

According to the previous theorem there is a q.c. homeomorphism H which conjugates the maps R_0 and R_λ on their postcritical sets if the maps f_0 and f_λ are conjugate. By pulling forward we can find a q.c. homeomorphism $\tilde{H}$ which is a conjugacy of the maps F_0 and F_λ on their postcritical sets.

Having this map $\tilde{H}$ we can proceed with the proof exactly in the same way as in Section 4.1. Indeed, we can construct a sequence of q.c. homeomorphisms H^k and take a subsequence converging to $\hat{H}$. If the map F_0 is nonrenormalizable, then the Julia set of F_0 has zero Lebesgue measure. The proof of this fact is given in Appendix 5.4. Thus, we can again conclude that $h_\lambda = \hat{H}$ and therefore $G_\lambda = F_\lambda$ if F_0 is combinatorially equivalent to F_λ.

4.3. *The case of a Misiurewicz map.* Finally, let us consider the case when f_0 is a Misiurewicz map.

LEMMA 4.5. *Let $f_\lambda : X \hookleftarrow$ be an analytic regular family of analytic unimodal maps with a nondegenerate critical point, $\lambda \in \Omega \subset \mathbb{R}^N$ where Ω is an open set. Suppose that the map f_{λ_0} does not satisfy Axiom* A *and that the critical point of f_{λ_0} is nonrecurrent. Then there is a neighborhood Ω' of λ_0 such that the set $\Upsilon_{\lambda_0} \cap \Omega'$ is an analytic variety.*

Since the critical point of f_0 is nonrecurrent, there exists a neighborhood U of c_0 such that $f_0^n(c_0) \notin U$ for all $n > 0$, where c_0 is a critical point of f_0. Let Σ_0 be a set of points which do not belong to the basin of attraction and whose forward orbits under iterates of f_0 do not enter U. Obviously, $f_0(c_0) \in \Sigma_0$ which is a closed set and does not contain neutral periodic points because f_0 is a regular map and we have assumed that the iterates of the critical point do not converge to a periodic attractor. Due to Mañè's theorem Σ_0 is a hyperbolic set and there exists a neighborhood $D \subset \mathbb{C}^N$ of 0 such that when λ is in D, f_λ has a hyperbolic set Σ_λ close to Σ_0 and the dynamics of f_λ on Σ_λ is conjugate to the dynamics of f_0 on Σ_0. Thus there exists a homeomorphism $h_\lambda : \Sigma_0 \to \Sigma_\lambda$. The set Σ_λ depends holomorphically on λ. Indeed, the periodic points in Σ_λ depend holomorphically on λ and they are dense in Σ_λ. Applying the λ-lemma we can conclude that for fixed z the map $h_\lambda(z)$ is holomorphic.

The maps f_0 and f_λ are combinatorially equivalent for some $\lambda \in D \cap \mathbb{R}^N$, if and only if $h_\lambda(f_0(c_0)) = f_\lambda(c_\lambda)$. The last equation is analytic with respect to λ; hence its solution is an analytic variety.

4.4. *Density of Axiom* A *in regular families.* Now we finish the proof of Theorem C.

First let us consider the case $N = 1$. Suppose that f_{λ_0} does not satisfy Axiom A and that the set Υ_{λ_0} contains infinitely many points in Y. Since Υ_{λ_0} is an analytic variety, it is an open set. However, from kneading theory we know that this set of combinatorially equivalent maps should be closed. We have arrived at a contradiction and hence the set Υ_0 has only finitely many points.

Now we shall prove that Axiom A maps are dense in Ω. We have already shown that if the iterates of the critical point of some map f_{λ_0} do not converge to a periodic attractor, then one can perturb this map within the family f_λ to some other map which is not combinatorially equivalent to f_{λ_0}. The kneading invariant changes continuously with λ; hence there is a map f_{λ_1} in the family close to f_{λ_0} such that the iterates of its critical point converge to some periodic attractor. If this attractor is hyperbolic, we are done because then there are no neutral periodic orbits and the map is an Axiom A map. The other case is that the attractor is a neutral periodic orbit. The multiplier of this periodic orbit is an analytic function with respect to λ; hence either there are maps in the family f_λ close to f_{λ_1} which do not have a neutral periodic orbit of the same period or such a neutral periodic orbit exists for all $\lambda \in \Omega$. In the former case we can find a map close to f_{λ_1} such that the iterates of its critical point converge to a hyperbolic periodic orbit (this orbit appears after a bifurcation of the neutral periodic orbit), and this map is an Axiom A map. The latter case is impossible because in this case the iterates of the critical point should converge to this neutral periodic orbit for all maps in the family and hence all maps in the family would be combinatorially equivalent.

4.5. *Construction of a regular family.* Now we are going to show how to derive Theorem A from Theorem C and first we will study some properties of regular maps.

LEMMA 4.6. *Any regular map $f \in C^3$ with a recurrent critical point has its neighborhood in the space of C^3 unimodal maps consisting of regular maps.*

$\lhd$ Since f is regular and its critical point is recurrent, the map f has no neutral periodic points. Consider a nice interval I_f around the critical point such that the first return map to $f(I_f)$ has negative Schwarzian derivative (see Theorem 1.2). It can be easily shown that if a map g is C^3 close to f, then for the map g there is a nice interval I_g close to I_f such that the first return map of g to $g(I_g)$ has negative Schwarzian derivative as well. Let J be an interval containing the critical point and let the interval I_f strictly contain J. The set of points whose iterates under the map f never enter the interval J is a union of some hyperbolic set, periodic attractors and points whose iterates converge to the periodic attractors. If g is C^3 close enough to f, then the interval I_g will contain J and the hyperbolic set and its periodic attractors persist. In this case the map g is regular. Indeed, if g has a neutral periodic point, then the orbit of this point necessarily passes through the interval $g(I_g)$. The first return map of g to $g(I_g)$ has negative Schwarzian derivative, and, hence, iterates of the critical point have to converge to this neutral point (this is a standard fact, see [Sin]). $\rhd$

The set of unimodal maps in $C^\omega(\Delta)$ which have a neutral periodic orbit of period K is an analytic variety of codimension 1; thus the complement of this set is open and dense in $C^\omega(\Delta)$. The set of maps which do not have neutral periodic orbits is equal to the intersection of all such complements for $K = 1, 2, \ldots$. Due to the Baire theorem this set is dense in $C^\omega(\Delta)$ as well. Thus we have proved the following lemma:

LEMMA 4.7. *The set of regular maps is dense in the space of unimodal maps $C^\omega(\Delta)$.*

Proof of Theorem A. We will show that any regular map with a recurrent critical point can be included in a nontrivial analytic family of regular analytic unimodal maps. This will imply Theorem A. Indeed, since the regular maps are dense in $C^\omega(\Delta)$ we can first perturb the given map to a regular map, and then we can construct a nontrivial family of regular analytic maps and apply Theorem 3.1.

First notice that if the map we need to perturb is infinitely renormalizable, then we can take any nontrivial family passing through this map and apply the statement formulated in the remark after Theorem C; see also Section 4.1. In this way we can obtain a map close to the original map such that the iterates

of its critical point converge to some periodic attractor. If this map has neutral periodic points, it is easy to perturb it to a map which does not have neutral periodic orbits, however the iterates of its critical point still converge to a periodic hyperbolic attractor. Obviously, this will be an Axiom A map. The same arguments apply to the case when the map we need to perturb is a Misiurewicz map. Indeed, in Section 4.3 we have only used the regularity of the map f_0 itself and we have never used the regularity of other maps in the family. Thus, we have only to construct a perturbation of an analytic unimodal regular nonrenormalizable map with a quadratic recurrent critical point.

Now we are going to construct a perturbation of f. First, it will be only a C^3 perturbation.

For any $\epsilon > 0$, Theorem 3.1 gives a polynomial-like map $F_\epsilon : B_\epsilon \to A_\epsilon$ induced by f. Let A_ϵ^c be a connected component of A_ϵ containing the critical point and let $a_\epsilon = f(\partial(A_\epsilon^c \cap \mathbb{R}))$. The interval $f(B_\epsilon^c) \cap \mathbb{R}$ has two boundary points as well and we let b_ϵ be one of these boundary points which does not have two real preimages under f. Just to fix the notation let us assume that $a_\epsilon < b_\epsilon$, which corresponds to the case when the map f first increases and then decreases.

Let the function $p_{\epsilon,\lambda} : \mathbb{R} \to \mathbb{R}$ be given by the following formula:

$$p_{\epsilon,\lambda}(x) = \begin{cases} x, & \text{if } x < a_\epsilon \\ x + \lambda \frac{(x-a_\epsilon)^4}{(b_\epsilon - a_\epsilon)^4}, & \text{if } x \geq a_\epsilon. \end{cases}$$

One can easily see that this function is C^3. The perturbation of the map f will have the form $p_{\epsilon_0,\lambda} \circ f$ for some sufficiently small ϵ_0 given by the following lemma:

LEMMA 4.8. *There exist $\lambda_0 > 0$ and ϵ_0 (depending on f) such that the maps f and $p_{\epsilon_0,\lambda_0} \circ f$ are not conjugate and there exists an analytic family of polynomial-like maps $F_{\epsilon_0,\lambda} : B_{\epsilon_0,\lambda} \to A_{\epsilon_0}$ induced by $p_{\epsilon_0,\lambda} \circ f$, where $\lambda \in [0, \lambda_0]$, $F_{\epsilon_0,0} = F_{\epsilon_0}$, $B_{\epsilon_0,0} = B_{\epsilon_0}$.*

Before giving a proof of this simple lemma let us notice that though the map $p^\lambda_{\epsilon_0} \circ f$ is only C^3 and not analytic, it can induce a polynomial-like map because the perturbation is not analytic just at one point whose forward orbit never comes inside of A_{ϵ_0}.

$\lhd$ First of all we can extend the function p_ϵ to the complex plain by the following formula:

$$p_{\epsilon,\lambda}(z) = \begin{cases} z, & \text{if } \Re(z) < a_\epsilon \\ z + \lambda \frac{(z-a_\epsilon)^4}{(b_\epsilon - a_\epsilon)^4}, & \text{if } \Re(z) \geq a_\epsilon \end{cases}$$

This function is discontinuous along the line $\Re(z) = a_\epsilon$.

Fix small $\lambda_0 > 0$. Consider a polynomial-like map F_ϵ and let us see what happens to it when we perturb the map f.

Due to Theorem 3.1, we know that the interval (a_ϵ, b_ϵ) is disjoint from A_ϵ and that if $F_\epsilon|_{B_\epsilon^x} = f^n$, then $f^i(x) \notin A_\epsilon^c$ for $i = 1, \ldots, n-1$. This implies that if we perturb f by $p_{\epsilon,\lambda}$, then this will not affect the map F_ϵ outside of $A_\epsilon \setminus A_\epsilon^c$. Let $F_{\epsilon,\lambda}|_{B_\epsilon \setminus A_\epsilon^c} = F_\epsilon$.

Again due to Theorem 3.1 if $x \in B_\epsilon \cap A_\epsilon^c$, then the size of $f(B_\epsilon^x)$ is very small compared to $|b_\epsilon - a_\epsilon|$. Hence if ϵ is small enough, we have

$$f^{-1} \circ {p_{\epsilon,\lambda}}^{-1} \circ f(B_\epsilon^x) \subset A_\epsilon^c$$

for any $x \in B_\epsilon \cap A_\epsilon^c$, where $0 \le \lambda \le \lambda_0$. Let

$$B_{\epsilon,\lambda} = (B_\epsilon \setminus A_\epsilon^c) \bigcup \left(f^{-1} \circ p_{\epsilon,\lambda}^{-1} \circ f(B_\epsilon \cap A_\epsilon^c) \right).$$

As we have seen, $B_{\epsilon,\lambda} \subset A_\epsilon$ for $0 \le \lambda \le \lambda_0$. Finally let

$$F_{\epsilon,\lambda}(x) = f^{n-1} \circ p_{\epsilon,\lambda} \circ f(x),$$

where $x \in B_{\epsilon,\lambda}$ and n is such that

$$F_\epsilon\big|_{B_\epsilon^{(f^{-1} \circ p_{\epsilon,\lambda} \circ f(x))}} = f^n.$$

Notice that if $x \notin A_\epsilon^c$, then $F_{\epsilon,\lambda}(x) = F_\epsilon$.

Decreasing ϵ if necessary we can get the following: $f(c) \notin f(B_{\epsilon,\lambda_0})$. Indeed, we know that the ratio $\frac{|b_\epsilon - f(c)|}{|f(c) - a_\epsilon|}$ can be made arbitrarily small by decreasing ϵ, so that $p_{\epsilon,\lambda_0} \circ f(c) \notin f(B_\epsilon)$. Thus F_ϵ and F_{ϵ,λ_0} cannot be conjugate. $\triangleright$

Notice that the perturbation $p_{\epsilon_0,\lambda_0} \circ f$ of the map f is large even in the C^1 topology.

The family of polynomial-like maps $F_{\epsilon_0,\lambda}$ is not trivial: $F_{\epsilon_0,0}$ and F_{ϵ_0,λ_0} are not conjugate. To this family we can apply the results of Section 4.2 and conclude that there is $\lambda_1 \in (0, \lambda_0)$ such that the maps $F_{\epsilon_0,0}$ and $F_{\epsilon_0,\lambda}$ are not conjugate for any $\lambda \in (0, \lambda_1)$. Hence, the maps f and f_λ are not conjugate as well, where $f_\lambda = p_{\epsilon_0,\lambda} \circ f$ and $0 < \lambda < \lambda_1$.

We already know that the map f has a C^3-neighborhood consisting of regular maps. Let us denote this neighborhood by U. Taking a smaller neighborhood if necessary we can assume that U is convex. Take $\lambda_2 < \lambda_1$ so small that the maps f_λ belong to U for $0 < \lambda \le \lambda_2$. Approximate this map f_{λ_2} by some analytic map g in such a way that the map g also belongs to U and the maps g and f are not conjugate. Notice that all the maps of the family f_λ have a critical point which does not depend on λ and the map g can be chosen in such a way that the critical points of f and g coincide. Let $g_\lambda = \lambda g + (1-\lambda) f$, $\lambda \in [0, 1]$. Then g_λ is an analytic nontrivial family of analytic unimodal regular maps with nondegenerate critical point. Theorem C implies that for small λ the maps f and g_λ are not conjugate. It is also clear that f and g_λ are close in the $C^\omega(\Delta)$ topology for small λ. $\square$

5. Appendix

5.1. *Quasiconformal homeomorphisms.* In this section we will give a short overview of definitions and results connected with quasiconformal maps. For the details the reader can consult books [Ahl], [LV].

There are many different, equivalent definitions of the quasiconformal (q.c.) homeomorphism. We will use the following:

Definition 5.1. Let $U \subseteq \bar{\mathbb{C}}$ be a domain in the complex plane. The map $h : U \to h(U)$ is called a *quasiconformal* homeomorphism if

- h is an orientation preserving homeomorphism between the domains U and $h(U)$;
- The real part $\Re(h)$ and the imaginary part $\Im(h)$ of h are absolutely continuous on almost all verticals and almost all horizontals in the sense of Lebesgue;
- There exists a constant $k < 1$ such that for
$$\mu_h(z) = \frac{d_{\bar{z}} f(z)}{d_z f(z)}$$
one has
$$|\mu_h(z)| < k$$
for almost all $z \in U$ where $d_{\bar{z}} h = \frac{dh}{d\bar{z}}$ and $d_z h = \frac{dh}{dz}$.

The function μ_h is called the *Beltrami coefficient* of a q.c. homeomorphism h.

To the Beltrami coefficient μ one can associate a field of infinitesimal ellipses. The eccentricities of these ellipses are given by $\frac{1+|\mu(z)|}{1-|\mu(z)|}$ and the directions of the major axes are given by $\sqrt{\mu(z)}$.

If f is a holomorphic map, we can pull back this field of ellipses even if f is not injective. This pullback we will denote as $f^*\mu$ which is equal to
$$(f^*\mu)(z) = \mu(f(z)) \frac{\overline{d_z f(z)}}{d_z f(z)}.$$

Here is a list of theorems to be used later on.

Theorem 5.1 (measurable Riemann mapping theorem). *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function such that $|\mu| < k < 1$ almost everywhere. Then there exists a unique* q.c. *homeomorphism $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ whose Beltrami coefficient is μ and which is normalized such that $h(0) = 0$, $h(1) = 1$ and $h(\infty) = \infty$.*

Theorem 5.2 (Ahlfors-Bers theorem). *Let $\Lambda \subset \mathbb{C}^n$ be an open set and $\mu : \mathbb{C} \times \Lambda \to \mathbb{C}$ be a measurable function satisfying:*

- $|\mu(z,\lambda)| < k < 1$ *for all* $\lambda \in \Lambda$ *and for almost all* $z \in \mathbb{C}$*;*
- *The map* $\lambda \mapsto \mu(z,\lambda)$ *is holomorphic in* λ *for almost all* $z \in \mathbb{C}$.

Then there exists a unique function $H : \mathbb{C} \times \Lambda \to \mathbb{C}$ *such that*

- $H(0,\lambda) = 0$, $H(1,\lambda) = 1$, $H(\infty,\lambda) = \infty$;
- *For fixed* $\lambda \in \Lambda$ *the map* $z \mapsto F(z,\lambda)$ *is a* q.c. *homeomorphism whose Beltrami coefficient is* $\mu(\cdot,\lambda)$;
- *The map* $\lambda \mapsto F(z,\lambda)$ *is holomorphic for almost every* z.

The first version of the next theorem appeared in [MSS] and after it was generalized several times: [BR], [Slo].

Theorem 5.3 (λ-lemma). *Let* $Z \subset \bar{\mathbb{C}}$ *be a set*, D *be an open unit disk in the complex plane and let* $h : Z \times D \to \bar{\mathbb{C}}$ *satisfy the following conditions*:

- $h(z,0) = z$ *for any* $z \in Z$;
- *For fixed* $z \in Z$ *the function* $\lambda \mapsto h(z,\lambda)$ *is holomorphic for* $\lambda \in D$;
- *For fixed* $\lambda \in D$ *the map* $z \mapsto h(z,\lambda)$ *is injective for all* $z \in Z$.

Then there exists $H : \bar{\mathbb{C}} \times D \to \bar{\mathbb{C}}$ *such that*

- $H(z,\lambda) = h(z,\lambda)$ *for* $\lambda \in D$ *and* $z \in Z$;
- $H(z,0) = z$ *for* $z \in \bar{\mathbb{C}}$;
- *For fixed* $z \in \bar{\mathbb{C}}$ *the function* $\lambda \mapsto H(z,\lambda)$ *is holomorphic for* $\lambda \in D$;
- *For fixed* $\lambda \in D$ *the map* $z \mapsto H(z,\lambda)$ *is a* q.c. *homeomorphism*;
- *For almost every* $z \in \bar{\mathbb{C}}$ *the Beltrami coefficient of* H *depends holomorphically on* λ.

Since the Beltrami coefficient of a q.c. homeomorphism is not defined everywhere we have to clarify the last item in the previous theorem. We say that the Beltrami coefficient depends holomorphically on λ for almost every z if there is a function $\mu(z,\lambda)$ such that for almost every z the function $\lambda \mapsto \mu(z,\lambda)$ is holomorphic and for fixed λ the equality $\mu(z,\lambda) = \mu_{H(\lambda,\cdot)}(z)$ holds almost everywhere.

Theorem 5.4 (Compactness of the set of q.c. homeomorphisms). *If* H *is a family of* q.c. *homeomorphisms of* $\bar{\mathbb{C}}$ *whose Beltrami coefficients are uniformly bounded by a constant* $k < 1$, *then any sequence in* H *has a subsequence which converges uniformly and the limit either a constant or a* q.c. *homeomorphism whose Beltrami coefficient is bounded by* k.

THEOREM 5.5. *If f is holomorphic, then $\mu_{f\circ h} = \mu_h$ and $\mu_{h\circ f}(z) = \mu_h(f(z))\frac{\overline{d_z f(z)}}{d_z f(z)}$.*

The real counterpart of q.c. homeomorphisms are quasisymmetric homeomorphisms of the real line.

Definition 5.2. The homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is called *quasisymmetric* if there is a constant $C > 0$ such that for any three points $x_{-1} < x_0 < x_1$ such that $x_0 - x_{-1} = x_1 - x_0$ the following inequality holds:

$$C^{-1} < \frac{|h(x_1) - h(x_0)|}{|h(x_0) - h(x_{-1})|} < C.$$

The following theorem describes relations between quasiconformal and quasisymmetric homeomorphisms:

THEOREM 5.6. *Let h^c be a quasiconformal homeomorphism of the complex plane such that its restriction h^r to the real line is a real function. Then this restriction is a quasisymmetric homeomorphism.*

If h^r is a quasisymmetric homeomorphism of the real line, then there is a quasiconformal homeomorphism $h^c : \mathbb{C} \to \mathbb{C}$ such that the restriction of h^c to the real line is h^r.

5.2. *The straightening theorem and geodesic neighborhoods.* One of the important applications of the measurable Riemann mapping theorem to holomorphic dynamical systems is the straightening theorem. Let $f : B \to A$ be a holomorphic proper 2-to-1 map where B and A are simply connected domains and A contains the closure of B. Such a map is called *quadratic-like*. Let $J(f) = \{z \in \mathbb{C} : f^i(z) \in U \text{ for all } i \le 0\}$. This set is called the *filled Julia set* of the quadratic-like map f. Douady and Hubbard proved the following result:

THEOREM 5.7 (The straightening theorem [DH]). *Let $f : B \to A$ be a quadratic-like map and d be the degree of f. Then there exists a quadratic map p, a neighborhood U of $J(f)$ such that $f : U \to f(U)$ is a quadratic-like map and there is a* q.c. *homeomorphism $h : f(U) \to p(h(U))$ which conjugates $f|_U$ and $p|_{h(U)}$.*

Let I be some interval on the real line. $\mathbb{C}_I$ will denote the domain $\mathbb{C}\backslash(\mathbb{R}\backslash I)$. Consider the Poincaré metric on the domain $\mathbb{C}_I$. It is clear that I is a geodesic in this metric. Denote the set of points whose distance in this metric to the interval I is less than l as $\tilde{D}_l(I)$.

Consider two circles S^- and S^+ centered at the points a^- and a^+ such that these points are symmetric with respect to the real line, and let these circles pass through the boundary points of the interval I and intersect the

real line at the angle $\phi < \frac{\pi}{2}$. Denote the intersection of the disks delimited by these circles as $D_\phi(I)$ and the union of these disks as $D_{\pi-\phi}(I)$. So, $D_\phi(I)$ is a lens as shown in Figure 1.

One can check that the domain $\tilde{D}_l(I)$ coincides with $D_\phi(I)$ for $l = \ln\tan(\frac{\pi}{4} + \frac{\phi}{4})$. (See [dMvS].)

If g is a univalent map of the domain $\mathbb{C}_I$, then it contracts the Poincaré metric. So we have the following lemma:

LEMMA 5.1. *Let $g : \mathbb{C}_I \to \mathbb{C}_{g(I)}$ be a univalent map and let $g(I) \subset \mathbb{R}$. Then for any interval $J \subseteq I$ and any ϕ,*

$$g(D_\phi(J)) \subseteq D_\phi(g(J)).$$

Obviously, if the interval I consists of positive real numbers, then the square root map is univalent on $\mathbb{C}_I$ and we can apply the previous lemma. Another case when we can use it, is a case of the Epstein class.

Definition 5.3. A map f belongs to the Epstein class if it is real analytic and any inverse branch $f^{-1} : I \to \mathbb{R}$ can be univalently extended to the domain $\mathbb{C}_I$; i.e., if J is an interval of the monotonicity of f and $I = f(J)$, then the map $f^{-1}|_I$ can be holomorphically extended and the extended map $f^{-1} : \mathbb{C}_I \to \mathbb{C}_J$ is univalent.

If an analytic map does not belong to the Epstein class, whenever the size of $D_\phi(I)$ is small compared to the size of the extension of f^{-1} to the complex plain, one can give an estimate of the shape of the pullback of $D_\phi(I)$. More precisely, the following lemma holds:

LEMMA 5.2 ([dFdM, Lemma 2.4]). *There exists a universal constant $\tau_3 > 1$ such that for any small $a > 0$ there exists $\theta(a) \in (0, \pi)$ satisfying $\theta(a) \to \pi$ and $a/(\pi - \theta(a)) \to 0$ as $a \to 0$ such that the following holds. Let $F : D \to \mathbb{C}$, where D is a unit disk, be univalent and symmetric with respect to the real line, and assume that $F(0) = 0$, $F(a) = a$. Then for all $\phi \in (0, \theta(a))$,*

$$F(D_\phi([0, a])) \subset D_{(1+a^{\tau_3})\phi}([0, a]).$$

5.3. *Construction of the holomorphic box mapping.* Following a suggestion of the referee we include an outline of the proof of Theorem 2.3 here. This theorem was proved in [LvS, Th. C] in the case of maps of the form $x \mapsto x^l + c$ where l is even and c is real. To generalize the result of [LvS] we will follow the proof given in Section 14 of [LvS]. We will also use the notation of that paper (though the author of the present paper thinks that it is slightly illogical) even if it is different from what we have used above. Though we will not give proofs

of lemmas if they are identical to [LvS] we will try to keep the exposition self-contained. In what follows we will assume that f is nonrenormalizable since the renormalizable case appears in [LvS, Th. 11.1].

Given a unimodal map f we say that $g \in \mathcal{E}(T^0)$ if T^0 is a nice symmetrical interval around the critical point and $g : \cup_i T^1_i \to T^0$ where $\cup_i T^1_i$ is a collection of disjoint subintervals of T_0. Moreover, the following properties are satisfied:

- If $i \neq 0$, the map $g : T^1_i \to T^0$ is a diffeomorphism onto T^0 of the form $f^{j(i)}$;
- Denoting T^1_0 by T^1 we have that $g : T^1 \to T^0$ is a unimodal map of the form f^j, $g(\partial T^1) \in T^0$ and the range of the map $f^{j-1} : f(T^1) \to T^0$ can be extended to T^0;
- All iterates of the critical point under g are in $\cup_i T^1_i$.

Next we say that $g \in \mathcal{E}(T^0, T^{-1})$ if T^{-1} is a nice symmetrical interval containing T^0, $g \in \mathcal{E}(T^0)$ and the range of the map $f^{j(i)-1} : f(T^1_i) \to T^0$ can be extended to T^{-1} for all i.

We can define low, high and center returns for maps in $\mathcal{E}(T^0)$ in the same way we did it for first entry maps in Section 1.6.

Now we introduce a renormalization operator $\mathcal{R}$ for maps in $\mathcal{E}(T^0)$. Notice that this operator is different from the one used above.

First, we define $\mathcal{R}g$ in the case when g is a low return. In this case $\mathcal{R}g$ will be in $\mathcal{E}(T^0)$.

Let g be a low return. For any point $x \in \cup T^1_i$ we define $s(x)$ as a minimal nonnegative integer such that $g^{s(x)}(x) \notin T^1$ (thus for $x \notin T^1$ we have $s(x) = 0$). Then we define the intermediate renormalizations $\hat{\mathcal{C}}g$ by $\hat{\mathcal{C}}g(x) = g^{s(x)+1}(x)$ and $\mathcal{C}g$ by $\mathcal{C}g(x) = g(x)$ if $x \notin T^1$ and $\mathcal{C}g(x) = g^{s(x)}(x)$ if $x \in T^1$ (the definition of $\mathcal{C}g$ is given just to keep the same notation as in [LvS]; we will not use it).

Lemma 5.3. *If $g \in \mathcal{E}(T^0)$, then the map $\hat{\mathcal{C}}g$ is in $\mathcal{E}(T^0)$ as well.*

$\lhd$ It is easy to see that any noncentral branch of $\hat{\mathcal{C}}g$ is a diffeomorphism onto T^0. Let $\hat{T}^1 \subset T^1$ be a central domain of $\hat{\mathcal{C}}g$ and V be a domain of $\hat{\mathcal{C}}g$ such that $g(c) \in V$. Then $\hat{\mathcal{C}}g|_{\hat{T}^1} = \hat{\mathcal{C}}g|_V \circ g|_{\hat{T}^1}$. However, since the range of the map $f^{j-1} : \hat{T}^1 \to T^0$ can be extended to T^0, where $g|_{T^1} = f^j$, the interval V is contained in this range. Now using the fact that $\hat{\mathcal{C}}g|_V$ is a diffeomorphism onto T^0 we obtain that the range of the map $\hat{\mathcal{C}}g|_{\hat{T}^1}$ can be extended to T^0. $\rhd$

If $\hat{\mathcal{C}}g$ is a low return again, we can define $\hat{\mathcal{C}}^2 g$ (for the second intermediate renormalization the function s has to be defined with respect to $\hat{T}^1$) and so on. Let $\hat{T}^i$ be a sequence of central domains of $\hat{\mathcal{C}}^i g$ and let $\tilde{T}^1$ be the central domain

of $\mathcal{C}g$. Let $\tilde{s}$ be minimal nonnegative number such that $\hat{\mathcal{C}}^{\tilde{s}}g(\hat{T}^{\tilde{s}})\cap\tilde{T}^1 \neq \emptyset$. Then the renormalization of g is $\mathcal{R}g = \hat{\mathcal{C}}^{\tilde{s}}g$. As a consequence of the previous lemma we obtain that $\mathcal{R}g \in \mathcal{E}(T^0)$.

Now let g be a high return. Let x be an orientation preserving fixed point of $g|_{T^1}$ and z_1 be a boundary point of T^1 such that x is between c and z_1. Take preimages $z_2, z_3, \dots$ of z_1 along the branch $g|_{[z_1,c]}$. Let U_k be an interval with boundary points z_k and the point symmetrical to z_k and choose $k \geq 0$ minimal such that $g(U_k) \supset U_k$. The map f is not renormalizable, hence k exists. Denote U_k by V^1. For $x \notin V^1$ define $\tilde{\mathcal{W}}g(x)$ by $\tilde{\mathcal{W}}g(x) = g^j(x)$ where j is minimal such that $x \notin U_j$. For $x \in V^1$ let $\tilde{\mathcal{W}}g$ be the first return map of g to V^1. Finally, let $\mathcal{W}g = \tilde{\mathcal{W}}g|_{V^1}$.

In [LvS, Lemma 14.1] it is proved that if $g \in \mathcal{E}(T^0)$ and g is a low return, then $\mathcal{W}g \in \mathcal{E}(V^1, T^0)$.

Lemma 5.4. *Suppose that $g \in \mathcal{E}(T^0)$ is a first return map of f to T^0 and that $\hat{\mathcal{C}}^i g$ is a low return for $i = 0, \dots, m-1$. Let U be a domain of $\hat{\mathcal{C}}^m g$ and let $\hat{\mathcal{C}}^m g|_U = f^n$. Then the orbit $f(U), f^2(U), \dots, f^m(U)$ has intersection multiplicity at most $m+1$.*

Here we say that a collection of intervals has *intersection multiplicity* k if any point is covered by not more than k intervals from this collection.

This lemma can be proved easily by induction.

Lemma 5.5. *Suppose that $g \in \mathcal{E}(T^0)$ is a first return map of f to T^0, that $\hat{\mathcal{C}}^i g$ is a low return for $i = 0, \dots, m-1$ and let $\hat{T}^i$ be a central domain of $\hat{\mathcal{C}}^i g$. Then the first return map of $\hat{\mathcal{C}}^m g$ to T^m coincides with the first return map of f to T^m.*

Moreover, if $\hat{\mathcal{C}}^m g$ is a high return, then $\mathcal{W} \circ \hat{\mathcal{C}}^m g \in \mathcal{E}(V^1, T^0)$ is a first return map of f to the interval V^1.

$\lhd$ We will prove by induction with respect to m that the first return map of $\hat{\mathcal{C}}^m g$ to any nice interval U contained in $\hat{T}^m$ is the first return map of f to U.

Let $x \in U$. Let R be the first return map of $\hat{\mathcal{C}}^{m-1}g$ to U. By the induction assumption R coincides with the first return map of f to U. Let $R(x) = (\hat{\mathcal{C}}^{m-1}g)^n(x)$. Then $(\hat{\mathcal{C}}^{m-1}g)^{n-1}(x) \notin \hat{T}^{m-1}$ because by the construction of $\hat{\mathcal{C}}^{m-1}g$ we have $\hat{\mathcal{C}}^{m-1}g(\hat{T}^{m-1}) \cap U = \emptyset$. Thus $R(x)$ can be written as

$$\begin{aligned} R(x) &= (\hat{\mathcal{C}}^{m-1}g)^n(x) \\ &= (\hat{\mathcal{C}}^{m-1}g|_{T^0\setminus\hat{T}^{m-1}} \circ (\hat{\mathcal{C}}^{m-1}g|_{\hat{T}^{m-1}})^{s_l}) \\ &\quad \circ \dots \circ (\hat{\mathcal{C}}^{m-1}g|_{T^0\setminus\hat{T}^{m-1}} \circ (\hat{\mathcal{C}}^{m-1}g|_{\hat{T}^{m-1}})^{s_1})(x) \\ &= (\hat{\mathcal{C}}^m g)^l(x), \end{aligned}$$

where $s_i \geq 0$, $i = 1, \dots, l$. Therefore, R is the first return map of $\hat{\mathcal{C}}^n g$ to U.

The case of the high return can be treated in the same way. ▷

Lemma 5.6. *Let f be a C^3 unimodal map with a nondegenerate recurrent critical point. For any $\epsilon > 0$ there exists $\tau < 1$ such that if T^{-1} is a sufficiently small nice interval, $g \in \mathcal{E}(T^0, T^{-1})$, T^{-1} is an ϵ-scaled neighborhood of T^0, $\hat{\mathcal{C}}^i g$ is a low return for $i = 0, \ldots, m-1$, T^i is a central domain of $\mathcal{R}^i g$, then*

$$\frac{|T^i|}{|T^0|} < \tau^i$$

where $i = 1, \ldots, m$.

◁ The standard cross-ratio estimate yields the fact that for any $\epsilon > 0$ there is $K < 1$ such that if T_j^1 is a domain of g, then $\frac{|T_j^1|}{|T^0|} < K$. Applying the standard cross-ratio estimate once again we obtain the required inequality. ▷

The next three lemmas are a version of Lemma 14.4 of [LvS] broken into three parts and adapted to our case.

Lemma 5.7. *Let f be a C^3 unimodal map with a nondegenerate recurrent critical point. For any $\epsilon > 0$, $\tau > 0$ there exists N such that if T^{-1} is a sufficiently small nice interval which is an ϵ-scaled neighborhood of T^0, $g \in \mathcal{E}(T^0, T^{-1})$ is a first return of f to T^0, $\hat{\mathcal{C}}^i g$ is a low return for $i = 0, \ldots, N$, $\hat{T}^N$ is a central domain of $\hat{\mathcal{C}}^N g$, R is the first return map of f to $\hat{T}^N$, U is a central domain of R, $R|_U = f^j$, then the range of the map $f^{j-1} : f(U) \to T^N$ can be extended to T^0. Moreover, if W is a connected component of the preimage $f^{-j+1}(T^N)$ containing $f(U)$, then*

$$\frac{|W \setminus f(U)|}{|f(T^N)|} < \frac{1}{2}.$$

◁ First, we notice that $f^j(\partial T^N)$ is not in the interior of T^0. Indeed, due to Lemma 5.5, R is a first return map of $\hat{\mathcal{C}}^N g$ to $\hat{T}^N$, so that $f^j = f^{j_1} \circ \hat{\mathcal{C}}^N g$ for some $j_1 \geq 0$. Now, $\hat{\mathcal{C}}^N(\partial T^N) \in \partial T^0$ and T^0 is a nice interval; hence $f^j(\partial T^N) \notin \operatorname{int} T^0$.

Next, by standard arguments (e.g. see Lemma 1.1) we obtain the extension of the range of $f^{j-1} : f(U) \to T^N$.

The required inequality can be obtained by the same estimate as in Lemma 3.1. Note that we can use this estimate because the preimage of one of the boundary points of T^0 by f^{-j+1} is in the closure of $f(T^N)$. ▷

Lemma 5.8. *Let f be a C^3 unimodal map with a nondegenerate recurrent critical point. For any N and $\epsilon > 0$ there is $\sigma \in (0,1)$ such that if T^{-1} is a sufficiently small nice interval, $g \in \mathcal{E}(T^0, T^{-1})$ where T^{-1} is an ϵ-scaled*

neighborhood of T^0, $\mathcal{R}^i g$ *is a low return for* $i = 0, \ldots, k-1$ *where* $k < N$, $\mathcal{R}^k$ *is a high return,* T^{k+1} *is a central domain of* $\mathcal{R}^k g$, $\mathcal{R}^k g|_{T^{k+1}} = f^j$, A *is a connected component of the preimage* $f^{-j+1}(T^0)$ *containing* $f(T^{k+1})$, *and*

$$\frac{|T^0|}{|V^1|} < (1-\sigma)^{-1}\frac{|T^{-1}|}{|T^0|},$$

then

$$\frac{|A \setminus f(T^{k+1})|}{|f(T^0)|} < 1 - \sigma.$$

$\lhd$ The proof of this lemma is nearly identical to the proof of the second assertion of Lemma 14.4 of [LvS] with slight modifications.

All combinatorial properties of unimodal maps used in that proof obviously hold in our case. In particular, Lemma 14.2 holds. The estimates of that proof also hold with some spoiling factors close to one. Namely, the first first spoiling factor appears in inequality 14.2, which in our case would look like:

$$\frac{|R|}{|I|} > C\mu_{i+1}\frac{|f(R)|}{|f(I)|}$$

where C is a constant close to 1 if T^0 is small. In the same fashion such spoiling factors appear in other inequalities there. They will start accumulating as we have more and more low returns. Thus, inequality 14.7 will have the form

$$\frac{|R'|}{|I'|} > C^r(1-\epsilon(\sigma))\left(\frac{|T^{k+1}|}{|T^0|}\right)^2\left(\frac{|T^0|}{|T^1|}\right)^\tau.$$

Here in our notation $R' = f(T^{k+1})$ and $I' = A \setminus f(T^{k+1})$.

The number of low returns is bounded by N, hence $r \le N$ is also bounded. If the interval T^0 is small enough, the constant C can be made as close to 1 as we want. Therefore, we get the same estimate 14.8

$$\frac{|R'|}{|I'|} > \kappa\left(\frac{|T^{k+1}|}{|T^0|}\right)^2$$

where $\kappa > 1$ is some constant. The rest is the same as in [LvS]. $\rhd$

Lemma 5.9. *Let* f *be a* C^3 *unimodal map with a nondegenerate recurrent critical point. There is* $\epsilon > 0$ *such that if* $g \in \mathcal{E}(T^0, T^{-1})$ *is a first return map of* f *to a sufficiently small interval* T^0, T^{-1} *is an* ϵ*-scaled neighborhood of* T^0, $g|_{T^1} = f^j$, W *is a connected component of the preimage* $f^{-j+1}(T^0)$ *containing* $f(T^1)$, *then*

$$\frac{|W \setminus f(T^1)|}{|f(T^0)|} < \frac{1}{2}.$$

This proof is identical to the proof of Lemma 5.7.

Proof of Theorem 2.3. Let f be a real-analytic nonrenormalizable unimodal map with nondegenerate recurrent critical point. After some analytic change of the coordinate we can assume that $f = \hat{f}(x^2)$ where $\hat{f}$ is a real-analytic diffeomorphism. Let Ω be a complex neighborhood of the image of f such that $\hat{f}^{-1}$ is univalent on Ω.

We know that there is a sequence of pairs of nice intervals $\{T_i^0, T_i^{-1}\}$ whose lengths tend to zero and such that the first return map of f to T_i^0 is in $\mathcal{E}(T_i^0, T_i^{-1})$. Moreover, there exists a constant $\epsilon > 0$ such that for all i the interval T_i^{-1} is an ϵ-scaled neighborhood of T_i^0 (see [Mar] or [Koz]).

For this ϵ, Lemma 5.7 gives N. There is also $\delta > 0$ such that if U is a domain of the first return map to T_i^0, then T_i^0 is a δ-scaled neighborhood of U. Fix some angle ϕ_0 slightly less than $\frac{\pi}{2}$. Then there is $\phi_1 \in (\phi_0, \frac{\pi}{2})$ such that $D_{\phi_1}(U) \subset D_{\phi_0}(T)$ if T is a δ-scaled neighborhood of U. Moreover, the modulus of $D_{\phi_0}(T) \setminus D_{\phi_1}(T)$ is bounded away from zero by some constant which depends only on δ.

Take a pair $\{T^0, T^{-1}\}$ from the sequence with such small intervals that Lemmas 5.6, 5.7, 5.8 and 5.9 start to work. Moreover, let T^0 be so small that if $f^n(U) \subset T^0$, $f^n|_U$ is a diffeomorphism, the intersection multiplicity of the orbit $f(U), \dots, f^n(U)$ is at most N, then $\sum_{i=1}^n |f^i(U)|^{\tau_3} < \log(\phi_1/\phi_0)$ and intervals $f^i(U)$ are small compared with the distance to $\partial\Omega$ as Lemma 5.2 requires. Here the constant $\tau_3 > 1$ is given by Lemma 5.2. Such a T^0 exists because of the absence of wandering domains; see also Lemma 5.2 in [Koz]. The last inequality implies $\prod_{i=1}^n (1 + |f^i(U)|^{\tau_3}) < \phi_1/\phi_0$.

Let g be the first return map to T^0. If g is a low return as well as $\hat{\mathcal{C}}^i g$ for $i = 1, \dots, N$, then let R be the first return to $\hat{T}^N$ where $\hat{T}^N$ is a central domain of $\hat{\mathcal{C}}^N g$. Let U be any noncentral domain of R and $R|_U = f^j$. Then the orbit of U is disjoint and $f^{-j}(D_{\phi_0}(T^N) \subset D_{\prod_{i=1}^j (1+|f^i(U)|^{\tau_3})\phi_0}(U) \subset D_{\phi_1}(U) \subset D_{\phi_0}(T^N)$. For the central domain U we have $f^{-j+1}(D_{\phi_0}(T^N)) \subset D_{\phi_1}(W)$ where W is as in Lemma 5.7. Pulling back $D_{\phi_1}(W)$ by f and using Lemma 5.7 we obtain $f^{-j}(D_{\phi_0}(T^N)) \subset D_{\phi_0}(T^N)$. Notice that all these pullbacks do not intersect because $\phi_1 < \frac{\pi}{2}$.

Now let $\mathcal{R}^i g$ be a low return for $i = 0, \dots, k-1$ and $\mathcal{R}^k g$ be a high return where $k < N$. By the construction of $\mathcal{R}^i g$ we know that $\mathcal{R}^k g = \hat{\mathcal{C}}^m g$ for some m. We can assume that $m < N$; otherwise we are in the previous case. Suppose we are in the setting of Lemma 5.8. Arguing as above we can construct a holomorphic box mapping whose real trace is $\mathcal{R}^k g$. Notice that we can use Lemma 5.2 because orbits of domains of $\mathcal{R}^k g$ have intersection multiplicity at most $m + 1 \le N$.

If Lemma 5.8 does not apply; i.e., the inequality $\frac{|T^0|}{|V^1|} \ge (1-\sigma)^{-1} \frac{|T^{-1}|}{|T^0|}$ is satisfied, we can consider the map $g_1 = \mathcal{W} \circ \mathcal{R}^k g \in \mathcal{E}(V^1, T^0)$. If either Lemma 5.7 or Lemma 5.8 applies to g_1, we are done, otherwise we obtain a

map $g_2 \in \mathcal{E}(V^2, V^1)$ and so on. The ratio $\frac{|V^{i-1}|}{|V^i|} > (1-\sigma)^i \frac{|T^{-1}|}{|T^0|}$ tends to infinity. Thus, sooner or later we will have that the interval V^{i-1} is an ϵ-scaled neighborhood of V^i where ϵ is given by Lemma 5.9. In this case we can proceed exactly in the same way as before. □

Remark. We have not used the fact that the critical point is quadratic. So, one can remove the condition on the nondegeneracy of the critical point in Theorem 2.3. (Note that if f is real-analytic, the critical point is always nonflat.)

5.4. *Lebesgue measure of the Julia set.* The following result is proven in [Lyu1, Cor. 2]:

THEOREM 5.8 ([Lyu1]). *Let $F : B \to A$ be a polynomial-like map, where A consists only from one connected domain and $\partial B \cap \partial A = \emptyset$, and let F be a nonremormalizable map. Moreover, suppose that the critical point of F is recurrent. Then the Julia set $J = \{x \in B : F^n(x) \in B \,\forall n \geq 0$ of F has zero Lebesgue measure.*

Here, F nonrenormalizable means that F does not induce a quadratic-like map.

This theorem can be easily generalized to arbitrary polynomial nonrenormalizable maps:

THEOREM 5.9. *Let $F : B \to A$ be a polynomial-like map, the critical point of F be recurrent and let F be a nonremormalizable map. Then the Julia set J of F has zero Lebesgue measure.*

The proof of this statement is very similar to the proofs of similar statements in [LvS] and [Lyu1]. Following a suggestion of the referee it is included here.

◁ Consider two cases. First assume that the ω-limit set of c is minimal. Let $R : \hat{B} \to A^c$ be the first return map to A^c. The domain $\hat{B}$ can contain infinitely many connected components. However, there are only finitely many connected components which contain points of the orbit of the critical point. Indeed, since c is recurrent the ω-limit set contains the orbit of c; since $\omega(c)$ is minimal and does not contain points of the boundary of $\hat{B}$, thus $\omega(c) \subset \hat{B}$; the set $\hat{B}$ is open and $\omega(c)$ is compact; hence there are finitely many components of $\hat{B}$ which cover $\omega(c)$.

Let $\tilde{B}$ be a union of these finitely many components. Then we can apply Theorem 5.8 to the map $R|_{\tilde{B}}$. So the Julia set of $R|_{\tilde{B}}$ has zero Lebesgue measure, hence the Lebesgue measure of J is zero as well.

Now suppose that the ω-limit set of c is nonminimal. Then there is a point $a \in \omega(c)$ such that $c \notin \omega(a)$.

Denote $A_0 = A$, $A_1 = B$ and $A_k = F^{-k}(A)$. The map F is nonrenormalizable; hence sizes of domains in A_k shrink to zero. Let k_0 be such that $A^c_{k_0}$ does not contain points from the orbit of a and let k_1 be such that $A^c_{k_1}$ is compactly contained in $A^c_{k_0}$. Consider the first entry map $R : \hat{B} \to \hat{A}$, where $\hat{A} = A^c_{k_1}$. It is easy to see that if $x \in (\hat{B} \setminus \hat{A})$ then the range of R can be univalently extended to $A^c_{k_0}$ (compare Lemma 1.1). On the other hand, we also have the following property: let the map $R : \hat{B}^x \to \hat{A}$ have a univalent extension $\tilde{R} : \tilde{B} \to A^c_{k_0}$; then for any k either A^x_k contains $\tilde{B}$ or $\tilde{B}$ contains A^x_k.

Suppose that the Julia set J has positive Lebesgue measure. Let b be a density point of J such that $c \in \omega(b)$. Such a point always exists because the set of points whose ω-limit sets do not contain the critical point has zero Lebesgue measure. Given k let n_k be a minimal integer such that $F^{n_k}(b) \in A^a_k$. Such an integer n_k exists because $a \in \omega(c) \subset \omega(b)$. Since n_k is minimal there is a domain $U_k \ni b$ such that F^{n_k} maps U_k onto A^a_k either univalently or as 2-to-1. Then there is domain $V_k \ni b$ such that $R \circ F^{n_k}$ maps V_k onto $\hat{A}$ univalently or as 2-to-1. The range of this map can be extended to $A^c_{k_0}$ and again the extension map is at most 2-to-1. As $k \to \infty$ the size of V_k tends to zero. Since b is a density point of J and J is invariant the relative density of J in $\hat{A}$ is 1. The Julia set is closed, hence $\hat{A} \subset J$. This is impossible. $\triangleright$

The Warwick University, Coventry, United Kingdom
E-mail address: oleg@maths.warwick.ac.uk

References

[Ahl] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill Book Co., New York, 1978.

[BM1] A. Blokh and M. Misiurewicz, Dense set of negative Schwarzian maps whose critical points have minimal limit sets, *Discrete Contin. Dynam. Systems* **4** (1998), 141–158.

[BM2] ———, Collet-Eckmann maps are unstable, *Comm. Math. Phys.* **191** (1998), 61–70.

[BR] L. Bers and H. L. Royden, Holomorphic families of injections, *Acta Math.* **157** (1986), 259–286.

[dFdM] E. de Faria and W. de Melo, Rigidity of critical circle mappings. II, *J. A. M. S.* **13** (2000), 343–370.

[DH] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. École Norm. Sup.* **18** (1985), 287–343.

[dMvS] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer-Verlag, New York, 1993.

[GS] J. Graczyk and G. Świątek, Generic hyperbolicity in the logistic family, *Ann. of Math.* **146** (1997), 1–52.

[Jak] M. V. Jakobson, Smooth mappings of the circle into itself, *Mat. Sb.* **85** (**127**) (1971), 163–188.

[Koz] O. S. Kozlovski, Getting rid of the negative Schwarzian derivative condition, *Ann. of Math.* **152** (2000), 743–762.

[LV] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, New York, 1973.

[LvS] G. Levin and S. van Strien, Local connectivity of the Julia set of real polynomials, *Ann. of Math.* **147** (1998), 471–541.

[Lyu1] M. Lyubich, On the Lebesgue measure of the Julia set of a quadratic polynomial, Stony Brook IMS preprint 1991/10, 1991.

[Lyu2] ———, Geometry of quadratic polynomials: Moduli, rigidity and local connectivity, Stony Brook IMS preprint 93-9, 1993.

[Lyu3] ———, Combinatorics, geometry and attractors of quasi-quadratic maps, *Ann. of Math.* **140** (1994), 347–404.

[Lyu4] ———, Dynamics of quadratic polynomials I-II, *Acta Math.* **178** (1997), 185–297.

[Mar] M. Martens, Interval dynamics, Ph.D. thesis, Delft, 1990.

[MdMvS] M. Martens, W. de Melo, and S. van Strien, Julia-Fatou-Sullivan theory for real one-dimensional dynamics, *Acta Math.* **168** (1992), 273–318.

[MSS] R. Mañè, P. Sad, and D. Sullivan, On the dynamics of rational maps, *Ann. Sci. École Norm. Sup.* **16** (1983), 193–217.

[Shi] M. Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets, *Ann. of Math.* **147** (1998), 225–267.

[Sin] D. Singer, Stable orbits and bifurcation of maps of the interval, *SIAM J. Appl. Math.* **35** (1978), 260–267.

[Slo] Z. Slodkowski, Holomorphic motions and polynomial hulls, *Proc. A. M. S.* **111** (1991), 347–355.

[Sul1] D. Sullivan, The universalities of Milnor, Feigenbaum and Bers, in *Topological Methods in Modern Mathematics*, SUNY at Stony Brook, 1991, Proc. Symp. held in honor of John Milnor's 60th birthday, pp. 14–21.

[Sul2] D. Sullivan, Bounds, quadratic differentials, and renormalization conjecture, *A. M. S. Centennial Publ.* **2**, 1992, 417–466.

[Yoc] J.-C. Yoccoz, On the local connectivity of the Mandelbrot set, preprint, 1990.

(Received March 30, 1999)